PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

VOLUME VII.

SESSION 1888-89.

EDINBURGH:

1889.

PRINTED BY
JOHN LINDSAY, HIGH STREET, EDINBURGH.

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OF THE

EDINBURGH MATHEMATICAL SOCIETY.

SEVENTH SESSION, 1888-89.

First Meeting, 9th November 1889.

W. J. MACDONALD, Esq., M.A., F.R.S.E., President, in the Chair.

For this Session the following Office-Bearers were elected :—

President—Mr GEORGE A. GIBSON, M.A.

Vice-President—Mr A. Y. FRASER, M.A., F.R.S.E.

Secretary—Mr JOHN ALISON, M.A.

Treasurer—Rev. JOHN WILSON, M.A., F.R.S.E.

Editors of Proceedings—Messrs R. E. ALLARDICE, M.A., F.R.S.E., and WILLIAM PEDDIE, D.Sc., F.R.S.E.

Committee—Mr A. C. ELLIOTT, D.Sc., C.E.; Rev. NORMAN FRASER, B.D.; Mr W. J. MACDONALD, M.A., F.R.S.E., and Mr J. T. MORRISON, M.A., B.Sc.

On the relations between systems of curves which, together, cut their plane into squares.

By Professor TAIT.

If ρ be the vector of a corner of a square in one system, σ that in a system derived without inversion, we must obviously have

$$d\sigma = u\left(\cos\frac{\phi}{2} + k\sin\frac{\phi}{2}\right)d\rho\left(\cos\frac{\phi}{2} - k\sin\frac{\phi}{2}\right),$$

$$= u\{(i\cos\phi + j\sin\phi)dx - (i\sin\phi - j\cos\phi)dy\}, \quad \ldots \quad (1)$$

k being the unit-vector perpendicular to the common plane.

This requires that

$$\frac{d}{dy}\left\{u(i\cos\phi + j\sin\phi)\right\} = \frac{d}{dx}\left\{u(-i\sin\phi + j\cos\phi)\right\},$$

which gives the two equations

$$\frac{du}{dy}\cos\phi + \frac{du}{dx}\sin\phi = u\left(\sin\phi\frac{d\phi}{dy} - \cos\phi\frac{d\phi}{dx}\right),$$

$$\frac{du}{dy}\sin\phi - \frac{du}{dx}\cos\phi = u\left(-\cos\phi\frac{d\phi}{dy} - \sin\phi\frac{d\phi}{dx}\right),$$

or, in a simpler form,

$$\left.\begin{aligned}\frac{1}{u}\frac{du}{dx} &= \frac{d\phi}{dy},\\ \frac{1}{u}\frac{du}{dy} &= -\frac{d\phi}{dx}.\end{aligned}\right\} \quad \ldots \quad \ldots \quad (2).$$

Eliminating ϕ and u separately, we have

$$\frac{d^2\log u}{dx^2} + \frac{d^2\log u}{dy^2} = 0,$$

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0.$$

Thus

$$\left.\begin{aligned}\log u &= C_1\\ \phi &= C_2\end{aligned}\right\} \quad \ldots \quad \ldots \quad (3)$$

represent associated series of equipotential, and current, lines in two dimensions.

Assuming any lawful values for the members of (2) we obtain u and ϕ, and thence, by integration of (1), σ is given in terms of ρ.

Thus $$\sigma = i\xi + j\eta,$$

where ξ and η are known functions of x and y. From this x and y can be found in terms of ξ, η. Thus if

$$F_1(x, y) = A_1,\ F_2(x, y) = A_2 \quad \ldots \quad \ldots \quad (4)$$

be a pair of sets of curves possessing the required property, we

obtain at once another pair by substituting for x and y their values in terms of ξ, η. These may now be written as x, y, and the process again applied, and so on.

Thus, let the values of the pairs of equal quantities in (2) be 1, 0, respectively (which is obviously lawful), we have

$$u = \epsilon^x, \ \phi = y;$$

so that (1) becomes

$$d\sigma = \epsilon^x\Big((i\cos y + j\sin y)dx - (i\sin y - j\cos y)dy\Big),$$

and $$\sigma = \epsilon^x(i\cos y + j\sin y)$$

or $$\xi = \epsilon^x\cos y, \ \eta = \epsilon^x\sin y.$$

From these we have

$$x = \log\sqrt{\xi^2 + \eta^2}, \ y = \tan^{-1}\frac{\eta}{\xi};$$

or, using polar cöordinates for the derived series,

$$x = \log r, \ y = \theta.$$

[This is easily seen to be only a special case of (3) above.] Hence, by (4), another pair of systems satisfying the condition is

$$\mathrm{F}_1(\log r, \ \theta) = \mathrm{A}_1, \ \mathrm{F}_2(\log r, \ \theta) = \mathrm{A}_2.$$

This, of course, is only one of the simplest of an infinite number of solutions of the equation (1), which may be obtained with the greatest ease from (2).

If there is inversion, all that is necessary is to substitute ρ^{-1} for ρ, or $-\rho^{-1}d\rho\rho^{-1}$ for $d\rho$. But the necessity for this may be avoided by substituting for any pair of systems which satisfy the condition their electric image, which also satisfies it, and which introduces the required inversion.

The solution of this problem without the help of quaternions is interesting. Keeping as far as possible to the notation above, it will be seen that the conditions of the problem require that

$$\left(\frac{d\xi}{dx}dx + \frac{d\xi}{dy}dy\right)^2 + \left(\frac{d\eta}{dx}dx + \frac{d\eta}{dy}dy\right)^2 = u^2(dx^2 + dy^2)$$

whatever be the ratio $dx : dy$.

This gives at once

$$\left(\frac{d\xi}{dx}\right)^2 + \left(\frac{d\eta}{dx}\right)^2 = \left(\frac{d\xi}{dy}\right)^2 + \left(\frac{d\eta}{dy}\right)^2 = u^2,$$

$$\frac{d\xi}{dx}\frac{d\xi}{dy} + \frac{d\eta}{dx}\frac{d\eta}{dy} = 0.$$

From these the equations (2) can be deduced by introducing ϕ as an auxiliary angle.

The value of cos2π/17 expressed in quadratic radicals.*

By Professor STEGGALL.

Call $2\pi/17$ a, and let

$$x_1 = 2\cos a,\ x_2 = 2\cos 2a, \text{ etc.}$$

Then from trigonometry

$$x_1x_2 = x_1 + x_3,$$

$$x_1x_3 = x_2 + x_4, \text{ etc };$$

and
$$\begin{aligned} x_1x_2x_4x_8 &= \frac{\sin 16a}{\sin a} = \frac{\sin(17a - a)}{\sin a} = \frac{\sin(2\pi - a)}{\sin a} \\ &= -1 \\ &= (x_1 + x_3)(x_4 + x_5) \\ &= x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8. \end{aligned}$$

Now let
$$\begin{aligned} x_1x_4 &= x_3 + x_5 = y_1, \\ x_2x_8 &= x_6 + x_7 = y_2, \\ x_6x_7 &= x_1 + x_4 = y_3, \\ x_3x_5 &= x_2 + x_8 = y_4. \end{aligned}$$

Then x_1, x_4 are the roots of

$$x^2 - y_3x + y_1 = 0 \qquad \ldots \qquad \ldots \qquad (1)$$

and we require the four y's.

We have already seen that

$$y_1 + y_2 + y_3 + y_4 = -1;$$

and we have

$$\begin{aligned} (y_1 + y_2)(y_3 + y_4) &= (x_3 + x_5 + x_6 + x_7)(x_1 + x_2 + x_4 + x_8) \\ &= 4(x_1 + x_2 + x_3 + \quad \ldots \quad \ldots \quad + x_8) \\ &= -4. \end{aligned}$$

Therefore $y_1 + y_2$, $y_3 + y_4$ are the roots of

$$\xi^2 + \xi - 4 = 0 \qquad \ldots \qquad \ldots \qquad (2)$$

$\therefore$
$$y_1 + y_2 = \tfrac{1}{2}(-1 - /17),$$

(the negative root being taken, because

$$\begin{aligned} y_1 + y_2 &= x_3 + x_5 + x_6 + x_7 \\ &= x_2x_5 + x_5 + x_6, \end{aligned}$$

where x_2 is positive and x_5, x_6 are negative)

And since
$$y_1y_2 = x_1x_2x_4x_8 = -1$$

Therefore y_1, y_2 are the roots of

$$y^2 + \tfrac{1}{2}(1 + \sqrt{17})y - 1 = 0 \qquad \ldots \qquad \ldots \qquad (3)$$

---

* This paper is merely intended to show how the solution of this interesting case of the binomial equation may be exhibited in a form suitable for a course of Elementary Trigonometry.

Similarly y_3, y_4 are the roots of

$$y^2+\tfrac{1}{2}(1-\sqrt{17})y-1=0 \quad \ldots \quad \ldots \quad (4)$$

Then from equations (3) and (4) attending to the proper signs we have

$$y_1=\{-1-\sqrt{17}+\sqrt{(34+2\sqrt{17})}\}/4 \quad \ldots \quad (5)$$

$$y_3=\{-1+\sqrt{17}+\sqrt{(34-2\sqrt{17})}\}/4 \quad \ldots \quad (6)$$

Substituting these values in equation (1) and solving, we have finally

$$x_1=2\cos 2\pi/17$$

$$=\Big\{\sqrt{17}-1+\sqrt{(34-2\sqrt{17})}$$

$$+\sqrt{\{68+12\sqrt{17}+2(\sqrt{17}-1)\sqrt{(34-2\sqrt{17})}-16\sqrt{(34+2\sqrt{17})}\}}\Big\}/8.$$

A short notice of the additions to the Mathematical Theory of Heat since the transmission of Fourier's Memoir of 1811 to the French Academy.

By George A. Gibson, M.A.

What is here printed contains merely a list of the memoirs and treatises that may perhaps be found useful for one who wishes to trace the progress of the mathematical theory of heat beyond the stage at which Fourier left it. As discussions of the Fourier series and integrals occur in almost every treatise on the Integral Calculus, I have omitted reference to these. Similar considerations have led me to omit references to the discussion of differential equations, except where these specially dealt with the problem of the conduction of heat.

Poisson.—Mémoires sur la Distribution de la Chaleur dans les Corps Solides. (*Journal de l'École Polytechnique*, t. xii., cah. 19, 1823.)

There are two memoirs, the first of which was presented to the Institute in 1815, and the second in 1821.

Poisson.—Théorie Mathématique de la Chaleur. (Paris, 1835.)

The problems Poisson discusses are in the main those of Fourier generalised. The methods he gives of proving the possibility of the expansion of a function in a Fourier series, or in a series of spherical harmonics, are those usually given in English text-books. The treatise contains little of importance that is not to be found in the memoirs.

Laplace (*Additions aux Connaissances des Temps*, 1823; *Mécanique Celeste*, Livre xi.) treated before Poisson, the case of the sphere with arbitrary initial distribution.

Duhamel.—"Mémoire sur la méthode générale relative au mouvement de la chaleur dans les corps solides plongés dans des milieux, dont la température varie avec le temps. (*École Polyt. Journ.*, t. xiv., 1833.) It is proved that the solution for any value of the time can be found by a simple integration if a solution for the time $t=$ const. can be found.

Sturm.—"Mémoire sur les équations differentielles lineaires du second ordre." (*Liouv. Jour.*, t. i., 1836, pp. 106-186.)

Sturm.—"Mémoire sur les équations aux différences partielles." (*Liouv. Jour.*, t. i., 1836, pp. 373-444.)

Liouville.—"Demonstration d'un Theorème dû à M. Sturm." (*Liouv. Jour*, t. i., 1836, pp. 269-277.)

These memoirs discuss equations of the form $\frac{d}{dx}\left(\mathrm{K}\frac{dy}{dx}\right)+\mathrm{G}y=0$, K,G being functions of x and of an arbitrary parameter r, with the view of tracing the variation of the roots of $y=0$ due to variation of r. The applications to the theory of heat are very important.

Thomson.—In Sir William Thomson's "Mathematical and Physical Papers" are several important articles on the conduction of heat, reprinted from the original journals. Articles LXXII. and LXXIII., together with Article XIV. of "Papers on Electrostatics and Magnetism," seem to me of special interest. The methods of these articles have been recently applied to several interesting problems by

Hobson.—"Synthetical Solutions in the Conduction of Heat." (*Proc. London Math. Soc.*, 1888, pp. 279-294.)

Lamé.—In his three treatises, "Les Fonctions Inverses des Transcendantes et les surfaces Isothermes" (Paris, 1857), "Coordonnées Curvilignes" (1859), and "Théorie Analytique de la Chaleur" (1861), will be found Lamé's contributions to the subject. His methods differ very considerably from those of his predecessors, and, as Mathieu remarks, he pays too little attention to them; but while it is probable Lamé estimates the value of his own methods too highly, he has nevertheless made important additions to the mathematics of physics generally.

Mathieu.—"Mémoire sur le mouvement de la température dans le corps renfermé entre deux cylindres circulaires excentriques et

dans des cylindres lemniscatiques" (*Liouv. Journ.*, t. xiv., 1869). A discussion of cases in which the simple solution cannot be put in the form of a product of factors, each of which contains one, and only one, of the thermometric parameters.

Mathieu.—"Cours de Physique Mathématiques" (Paris, 1873). An excellent treatise, giving a good treatment of the conduction of heat in its present mathematical development.

Boussinesq.—"Sur les problemes des températures stationaires, &c." (*Liouv. Journ.*, t. vi., 1880). A short notice of Lamé's methods.

Neumann, C.—"Über das Gleichgewicht der Wärme und das der Electricät in einem Körper welcher von zwei nicht concentrischen Kugelflächen begrenzt wird" (*Crelle's Journ.*, Bd. LXII.). This is merely a notice of the methods used by the author in a book entitled "Allgemeine Lösung des Problems über den stationären Temperaturzustand eines homogenen Körpers welcher von irgend zwei nicht concentrischen Kugelflächen begrenzt wird" (Halle, 1862). The book itself I have not seen, but the method of solution seems ingenious, though complicated.

Tait.—"On Orthogonal Isothermal Surfaces" (*Edin. Trans.*, vol. xxvii., 1872). An investigation by quaternion methods of properties of isothermal surfaces.

On the conduction of heat in crystalline media, I may refer to the following.

Duhamel.—"Sur les équations générales de la propagation de la chaleur dans les corps solides dont la conductibilité n'est pas la même dans tous les sens" (*École Polyt. Journ.*, t. xiii., 1832).

Duhamel.—"Note sur les surfaces isothermes dans les corps solides dont la conductibilité n'est pas la même dans tous les sens" (*Liouv. Journ.*, t. iv., 1839).

Duhamel.—"Sur la propagation de la chaleur dans les cristaux" (*École Polyt. Journ.*, t. xix., 1848).

Stokes.—"On the conduction of heat in crystals" (*Camb. and Dub. Math. Journ.*, vol. vi., 1851). This article of Stokes establishes all Duhamel's results in a very ingenious, yet simple manner.

Lamé.—"Théorie Analytique de la Chaleur."

Boussinesq.—"Étude sur les surfaces isothermes et les courants de chaleur dans les milieux homogénes échauffés en un de leurs points" (*Liouv. Journ.*, t. xiv., 1869).

The following text-books present a good many of the methods to be found in the articles quoted above in a very useful form:—

Riemann's "Partielle Differentialgleichungen;" Heine's "Kugelfunctionen," vol. ii., pp. 302-332; Todhunter's "Functions of Laplace, Lamé, and Bessel;" Jordan's "Cours d'Analyse, vol. iii., chap. iii., Part iv.

The notice now given has, of course, no pretentions to being exhaustive; but it may perhaps serve a useful purpose in helping one to follow the development of the theory whose basis Fourier so thoroughly established.

Second Meeting, December 14th, 1888.

GEORGE A. GIBSON, Esq., M.A., President, in the Chair.

On the general equation of the second degree representing a pair of straight lines.

By DAVID MUNN, M.A.

Kötters synthetic geometry of algebraic curves—Part I, imaginary curves.

By Rev. NORMAN FRASER, M.A.

[See Index.]

Third Meeting, January 11th, 1889.

GEORGE A. GIBSON, Esq., M.A., President, in the Chair.

Note on a Formula in Quaternions.

By R. E. ALLARDICE, M.A.

The formula referred is the condition for the coplanarity of the extremities of four coinitial vectors; namely, if α, β, γ, δ, are the vectors, then

$$a\alpha + b\beta + c\gamma + d\delta = 0, \text{ where } a + b + c + d = 0.$$

(See Kelland and Tait's Quaternions, p. 62.)

In the first place, it may be pointed out that these equations do involve one condition on the four vectors α, β, γ, δ. For, since the vanishing of a vector involves three conditions, and only the three ratios of the four quantities a, b, c, d, are involved, it is always possible to determine these four quantities so that the condition $a\alpha + b\beta + c\gamma + d\delta = 0$ shall be satisfied. But, in general, the equation $a + b + c + d = 0$ will not be satisfied; and thus one condition is involved.

The object of this note is to point out the geometrical interpretation of the above condition, and to furnish an independent proof of the theorem in Solid Geometry that is involved in it.

Let the four points ABCD be coplanar; and let α, β, γ, δ, denote the vectors OA, OB, OC, OD. Take A′B′C′D′ on OA, OB, OC, OD, respectively, so that $OA' = a\alpha$, &c. Then, writing the condition for coplanarity in the form $a\alpha - b\beta + c\gamma - d\delta = 0$ where $a - b + c - d = 0$, we have from the latter equation $OA'/OA - OB'/OB + OC'/OC - OD'/OD = 0$. Also the first equation may be written $a\alpha - b\beta = d\delta - c\gamma$; and hence B′A′ is parallel and equal to C′D′, and therefore A′B′C′D′ is a parallelogram. Hence the condition for the coplanarity of A, B, C, D, is equivalent to the following proposition:—

If four straight lines OA, OB, OC, OD, are met by a plane in the points A, B, C, D, and another plane meets the straight lines in the points A′, B′, C′, D′, so that A′B′C′D′ is a parallelogram, then

$$OA'/OA - OB'/OB + OC'/OC - OD'/OD = 0.$$

Geometrical Proof.

Let the plane of the paper (fig. 1) be the plane of ABCD and let O be the point of concurrence of the four lines. A parallelogram is obtained by drawing a plane parallel to OE and OF. E′, E″, F′, F″ are the points where this plane meets the line EA, EC, FB, FD. Let fig. 2 represent the plane determined by AB and O.

Then $\quad OA'/OA = EE'/EA$ and $OB'/OB = EE'/EB$;

$\therefore \quad OB'/OB - OA'/OA = EE'.AB/EA.EB.$

Similarly $\quad OC'/OC - OD'/OD = EE''.\ CD/EC.ED.$

We have now to show that $EE'.AB/EA.EB = EE''.CD/EC.ED.$

Now, $EE'/EE'' = \sin CEK/\sin BEH = \sin GED/\sin GEA =$
$(LD/ED)/(AL/AE).$

This reduces the above relation to

$$\frac{LD.AB}{ED.EA.EB} = \frac{AL.CD}{EA.EC.ED};\ \text{that is}$$

$$LD.AB.EC = AL.CD.EB;$$

which is seen to follow from Ceva's Theorem on considering the triangle ADE and the point G.

The corresponding theorem *in plano* is :—

If a transversal ABC meets three concurrent lines OA, OB, OC, and A′, B′, C′, are points in these lines such that OA′B′C′ is a parallelogram, then

$$\mathrm{OA'/OA - OB'/OB + OC'/OC = 0};$$

a theorem which is very easily proved.

If we invert the four points A, B, C, D, of the theorem proved above into the points P, Q, R, S, taking O as centre and k as radius of inversion, we have

$$\mathrm{OA.OP} = k^2\,;\ \therefore\ \mathrm{OA} = k^2/\mathrm{OP}.$$

Substituting in the relation

$$\mathrm{OA'/OA - OB'/OB + OC'/OC - OD'/OD = 0},$$

we get $\quad \mathrm{OA'.OP - OB'.OQ + OC'.OR - OD'.OS = 0}.$

Hence if four points P, Q, R, S, lie on the same sphere with the point O and a plane cuts OP, OQ, OR, OS, in A′, B′, C′, D′, so that A′B′C′D′ is a parallelogram, then the above relation holds.

The condition that the extremities of four vectors lie on a sphere passing through the origin, may be written

$$\frac{a}{\alpha^2}.\alpha + \frac{b}{\beta^2}.\beta + \frac{c}{\gamma^2}.\gamma + \frac{d}{\delta^2}.\delta = 0, \text{ where } a+b+c+d=0\,;$$

or, $\quad p\alpha + q\beta + r\gamma + s\delta = 0, \text{ where } p\alpha^2 + q\beta^2 + c\gamma^2 + d\delta^2 = 0.$

On the number of elements in space.

By Rev. NORMAN FRASER, M.A.

On the solution of the equation $x^p - 1 = 0$ (p being a prime number).

By J. WATT BUTTERS.

[At the first meeting of this Session a paper was read on the value of cos $2\pi/17$, which evidently may be made to depend on the

solution of $x^{17}-1=0$.* The present paper is the outcome of a suggestion then made, that a sketch of Gauss's treatment of the general equation might prove interesting. To give completeness to the subject the necessary theorems on congruences have been prefixed. The convenient notation introduced by Gauss is here adopted; thus, when the difference between a and b is divisible by p, instead of writing $a = \mathrm{M}p + b$, we may write $a \equiv b \pmod{p}$, the value of M seldom being of importance. It is evident that if $a \equiv b$, then $na \equiv nb$, and $a^n \equiv b^n$, n being any positive integer, and the same modulus p being understood throughout. Also $a/n \equiv b/n$ provided n be prime to p. Other properties (similar to those of equations) are easily seen, but only the above are needed here.

Besides the *Disquisitiones Arithmeticæ* of Gauss, which was published in 1801, and of which there is a French translation, entitled, *Recherches Mathématiques*, the following, among others, have been consulted:—Legendre's *Théorie des Nombres*, Murphy's *Theory of Equations* (1839), two papers, by M. Realis, in *Nouvelles Annales de Mathématiques* (1843), Barlow's *Theory of Numbers* (1811). Other references will be found at the end.]

§ 1. If p be prime to a, then t can be found such that $a^t \equiv 1 \pmod{p}$, and $0 < t < p$.

Consider the series

$$a,\ a^2,\ \ldots\ a^{p-1} \qquad (1);$$
$$1,\ 2,\ \ldots\ p-1 \qquad (2).$$

Since p is prime to a, and therefore to each power of a, if the terms in (1) be divided by p, there can be no zero remainder. Hence either (a) the remainders will be all different, and therefore the same as (2), or (b) two at least will be the same. If (a) be true, the theorem follows directly; if (b), let $a^m \equiv a^n \pmod{p}$ (where $p > m > n$), then $a^{m-n} \equiv 1$ where $0 < m-n < p$.

Cor. 1. If $a^t \equiv 1$, $a^{t+1} \equiv a$, &c., *i.e.*, the remainders of the powers of a when divided by p will recur in groups of t terms. In symbols, $a^{rt+s} \equiv a^s$; or, if $m \equiv n \pmod{t}$ then $a^m \equiv a^n \pmod{p}$.

Cor. 2. If a^d be the lowest power of a which is $\equiv 1 \pmod{p}$, then the remainders got by dividing

$$1,\ a,\ a^2,\ \ldots\ a^{d-1} \qquad (3),$$

by p will be all different, and will be included in the series (2). In such a case a is said to *belong* to the exponent d.

* An elementary algebraic solution of this equation is given in *Knowledge*, vol. iii., p. 316 (1883).

§ 2. If p be a prime number which does not divide a, and a belong to the exponent d, then d is a factor of $p-1$.

Let the remainders got from the series (3) be

$$1,\ a,\ a',\ a'',\ \ldots\ (d \text{ terms}) \qquad (4).$$

If these (which by § 1. Cor. 2 are all different) include all the terms of the series (2), then $d = p-1$.

[a is then called a *primitive root* of p (Euler).]

If (4) is not the same as (2), let β be a term in (2) which is not in (4), and let the remainders of β, $a\beta$, $a'\beta$, $a''\beta$, ... be

$$\beta,\ \beta',\ \beta'',\ \ldots\ (d \text{ terms}) \qquad (5).$$

(a) No two terms in (5) are congruent, and (b) no term in (5) is congruent with a term in (4).

(a) For if $\beta a^m \equiv \beta a^n \pmod p$, then $a^m \equiv a^n$, which is impossible by § 1, Cor. 2.

(b) If $\beta a^m \equiv a^n$, then, according as $m >$ or $< n$, $\beta a^d \equiv a^{n+d-m}$ or $\beta \equiv a^{n-m}$; *i.e.* (since $a^d \equiv 1$), $\beta \equiv a^{d-(m-n)}$ or $\equiv a^{n-m}$, which is contrary to the hypothesis that β is not in (4).

If series (4) and (5) exhaust (2), then $2d = p-1$; if not, we may proceed in the same manner, always getting another group of d terms (such that none of the terms in all the groups are congruent) until (2) is exhausted, which must take place as $p-1$ is finite. We see, therefore, that $(p-1)/d$ is an integer.

[Cor. By raising each side of the congruence $a^d \equiv 1 \pmod p$ to the integral power $(p-1)/d$, we get $a^{p-1} \equiv 1 \pmod p$, which is Fermat's theorem.]

§ 3. Lemma:—If d, d', d'', ... be all the divisors of $p-1$ (including $p-1$ and unity) and if ϕd denote the number of integers not greater than d and prime to it*, then $\phi d + \phi d' + \ldots = p-1$.

If we multiply $(p-1)/d$ by each integer prime to d and not greater than it, we shall get ϕd integers, each not greater than $p-1$, and all unequal. Similarly, from d' we shall get $\phi d'$ integers, all unequal, and each $\not> p-1$. The integers in ϕd will also differ completely from those in $\phi d'$.

For if not we should have $m\dfrac{p-1}{d} = n\dfrac{p-1}{d'}$ where m is prime to d and n to d'. Consequently, $md' = nd$. We may suppose $m > n$, then since m is prime to d and divides nd it must divide n, which is impossible. Hence, from all the divisors d, d', &c., we shall get

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* Unity is considered as being prime to every number, itself included.

$\phi d + \phi d' + \ \dots$ different integers, each not greater than $p-1$, and hence comprised in the series (2). Further, each term in (2) will be found included in the $\phi d + \phi d' + \ \dots$ integers; for, let t be any term of that series and δ the G. C. M. of t and $p-1$, then $(p-1)/\delta$ will be a divisor to which t/δ is prime, and the product of $(p-1)/\{(p-1)/\delta\}$ by $t/\delta = t$. Hence $\phi d + \phi d' + \ \dots \ = p - 1$.

§ 4. Theorem: The number of integers less than p belonging to the exponent d is ϕd

If a be an integer belonging to d, then the terms in (3), or their remainders, are roots of $x^d \equiv 1 \pmod p$. Since this congruence cannot have more than d different roots and the above remainders are d in number and all different, it follows that the series (3) must contain all the integers belonging to d. Let ψd denote the number of them.

Let a^k be one of the series, then a^k does or does not belong to d, according as k is or is not prime to d.

1°. Suppose k prime to d and let $km \equiv 1 \pmod d$, then (§ 1. Cor. 1.) $a^{km} \equiv a \pmod p$. If possible, let $(a^k)^e \equiv 1$, where $e < d$; $a^{kme} \equiv 1$ and $\therefore$ $a^e \equiv 1$, which is contrary to the hypothesis that a belongs to d. Hence a^k belongs to d.

2°. Suppose k not prime to d and let δ be a common divisor. Since $kd/\delta \equiv 0 \pmod d$, $a^{kd/\delta} \equiv 1 \pmod p$, *i.e.*, $(a^k)^{d/\delta} \equiv 1$. Hence a^k does not belong to d.

Thus we have proved that if there be any integer belonging to d, there are as many as there are integers not greater than d and prime to it, *i.e.*, $\psi d = 0$ or $= \phi d$.

Now, evidently each term of the series (2) must belong to one of the divisors of $p-1$,

and hence $$\psi d + \psi d' + \psi d'' + \ \dots \ = p - 1 \qquad \text{(A)},$$
but $$\phi d + \phi d' + \phi d'' + \ \dots \ = p - 1 \qquad \text{(B)},$$
and since no term in (A) can exceed the corresponding term in (B), we must have $\psi d = \phi d$, &c.

Cor. This contains, as a particular case, the important theorem that every prime number has at least one primitive root. This amounts to saying that it is always possible to find an integer g, so that to each term of the series

$$1, g, g^2, \ \dots \ g^{p-2} \qquad (6)$$

there will be congruent, to the modulus p, one of the series

$$1, 2, 3, \ \dots \ p-1. \qquad (2)$$

Further: If λ be not divisible by p, then the series

$$\lambda, \lambda g, \lambda g^2, \ \dots \ \lambda g^{p-2} \qquad (6')$$

is congruent with the series (2).

§ 5. Now, we know from the Theory of Equations that if r denote any imaginary root of the equation $x^p - 1 = 0$, then all the roots of $X \equiv (x^p - 1)/(x - 1) = 0$ are given by

$$r,\ r^2,\ r^3,\ \dots\ r^{p-1} \tag{7}$$

Moreover, since $r^p = 1$, $r^{p+1} = r$, &c., and generally $r^{mp+a} = r^a$, we see that if $a \equiv b \pmod p$ then $r^a = r^b$. Hence, by § 4, Cor., instead of the series (7) we may use

$$r\lambda,\ r\lambda g,\ r\lambda g^2,\ \dots\ r\lambda g^{p-2}$$

to express the roots of $X = 0$.

To avoid the difficulty of printing the roots in this form, Gauss expresses them by the notation

$$[\lambda],\ [\lambda g],\ [\lambda g^2],\ \dots\ [\lambda g^{p-2}] \tag{8}$$

Evidently we have $[\lambda].[\mu]/ = [\lambda + \mu]$ and $[\lambda]^\mu = [\lambda\mu]$. Further, the roots $[\lambda g^m]$, $[\lambda g^n]$ will be identical or different according as m is congruent or is not congruent with n (mod $p - 1$).

§ 6. Since $p - 1$ is not a prime number, we may suppose $p - 1 = ef$. We may then write the roots as follows (putting $\lambda = 1$) :—

$$\begin{array}{lllll}
[1], & [g^e], & [g^{2e}], & \dots & [g^{(f-1)e}] \\
[g], & [g^{e+1}], & [g^{2e+1}], & \dots & [g^{(f-1)e+1}] \\
[g^2], & [g^{e+2}], & [g^{2e+2}], & \dots & [g^{(f-1)e+2}] \\
\cdot & \cdot & \cdot & \dots & \cdot \\
[g^{e-1}], & [g^{2e-1}], & [g^{3e-1}], & \dots & [g^{ef-1}].
\end{array} \tag{9}$$

Where the first column contains the first e roots, the second column the second e roots, and so on.

If the series (8) be extended indefinitely, we know that any $p - 1$ consecutive terms will denote the roots given by (8) itself. Hence, considering the mode of formation of the above table, we see that if any row be extended indefinitely it also will reproduce the same roots as are given by the row itself. Hence if, for brevity, we put $g^e = h$, and denote the sum of the roots in any row by (f, λ), we may also denote it by $(f, \lambda h)$, $(f, \lambda h^2)$, ... $(f, \lambda h^{f-1})$.

When we wish to speak of the roots in (f, λ) without expressing the idea of summation, we may speak of the *period* (f, λ). Periods containing the same number of roots are called *similar*.

Cor. 1. If (f, λ), (f, μ) denote similar periods, they will be identical, if they contain a common root. If μ is not divisible by p, then (f, μ) will be identical with one of the periods $(f, 1)$, (f, g), (f, g^2), ... (f, g^{e-1}). If $\mu \equiv 0 \pmod p$, then (f, μ) will be equal to f units. These results follow directly from the above table.

Cor. 2. If f be not a prime number, say, $=ab$, then any period (f, λ) may be written thus:—

$$\begin{array}{llll} [\lambda], & [\lambda h^a], & \dots & [\lambda h^{(b-1)a}] \\ [\lambda h], & [\lambda h^{a+1}], & \dots & [\lambda h^{(b-1)a+1}] \\ \cdot & \cdot & \cdots & \cdot \\ [\lambda h^{a-1}], & [\lambda h^{2a-1}], & \dots & [\lambda h^{ab-1}]; \end{array}$$

i.e., we may break up any period (f, λ) or (ab, λ) into smaller periods (b, λ), $(b, \lambda h)$, ... $(b, \lambda h^{a-1})$. If b have factors, the process may be repeated, and so on.

§ 7. If (f, λ), (f, μ) be two similar periods, then

$$(f, \lambda).(f, \mu) = (f, \lambda+\mu) + (f, \lambda h+\mu) + (f, \lambda h^2+\mu) + \dots + (f, \lambda h^{f-1}+\mu).$$

By § 6. $(f, \lambda) = (f, \lambda h) = (f, \lambda h^2) = \dots = (f, \lambda h^{f-1})$.

Hence $(f, \lambda).(f, \mu) = [\mu].(f, \lambda) + [\mu h].(f, \lambda h) + \dots + [\mu h^{f-1}].(f, \lambda h^{f-1})$

$$\begin{array}{lllll} =[\lambda+\mu] & +[\lambda h+\mu] & +[\lambda h^2+\mu] & +\dots+[\lambda h^{f-1}+\mu] \\ +[\lambda h+\mu h] & +[\lambda h^2+\mu h] & +[\lambda h^3+\mu h] & +\dots+[\lambda h^f+\mu h] \\ \dots & \dots & \dots & \dots \quad \dots \\ +[\lambda h^{f-1}+\mu h^{f-1}] & +[\lambda h^f+\mu h^{f-1}] & +[\lambda h^{f+1}+\mu h^{f-1}] & +\dots+[\lambda h^{2f-2}+\mu h^{f-1}] \end{array}$$

= (by adding the terms in each column together)

$$(f, \lambda+\mu) + (f, \lambda h+\mu) + (f, \lambda h^2+\mu) + \dots + (f, \lambda h^{f-1}+\mu).$$

Cor. 1. From the results of § 6, Cor. 1, we see that the above product may be put in the form

$$(f, \lambda).(f, \mu) = af + b(f, 1) + c(f, g) + d(f, g^2) + \dots + k(f, g^{e-1}) \quad (10)$$

where the coefficients a, b, ... k are known integers.

Cor. 2. The product of any number of similar periods can be expressed in the form (10).

Cor. 3. Hence any rational integral function of similar periods can be expressed in the same form.

§ 8. If $(f, \lambda) = n$, then any similar period (f, μ) can be expressed in the form $\qquad (f, \mu) = a + bn + cn^2 + \dots$

where a, b, c, ... are rational coefficients.

Let n, n', n'', ... denote the periods (f, λ), $(f, \lambda g)$, $(f, \lambda g^2)$,... as far as $(f, \lambda g^{e-1})$, with one of which, say n', (f, μ) must coincide (unless $\mu \equiv 0 \pmod p$, when $(f, \mu) = f$).

Since the sum of the roots of $X=0$ is -1 we have

$$0 = 1 + n + n' + n'' + \dots$$

and forming by § 7, Cor. 2, the valnes of n^2, n^3, ... n^{e-1} we get $e-2$ other equations

$$
\begin{aligned}
n^2 &= af \;\, + bn \;\, + cn' \;\, + dn'' \;\, + \; \ldots \\
n^3 &= a'f + b'n + c'n' + d'n'' + \; \ldots \\
n^4 &= a''f + b''n + c''n' + d''n'' + \; \ldots \\
\ldots \quad & \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots
\end{aligned}
$$

in which the coefficients are rational and independent of λ.

From these $e-1$ equations eliminate the $e-2$ quantities n'', n''',... and we will get an equation of the form $A + Bn + Cn^2 + \ldots + Mn^{e-1} + Nn' = 0$ (where A, B, ... N are integers and not all zero), which proves the theorem if N be not zero.

If we suppose that $N = 0$, then we get the equation

$$Mn^{e-1} + \ldots + Cn^2 + Bn + A = 0.$$

Now, since the $e-1$ equations from which this equation is deduced are all independent of λ, so will this equation be. It should therefore have the e roots $(f, 1)$, (f, g), (f, g^2), ... (f, g^{e-1}), but this is impossible since its degree is $e-1$; and therefore N cannot vanish.

[Gauss further considers the possibility of two of these roots being equal and there being therefore *apparently* only $e-1$ roots.]

Cor. If we form as above the values of n^2, n^3, ... n^e in terms of n, n', n'', ... and from the e equations eliminate the $e-1$ quantities n', n'', ... we shall get an equation of the *eth* degree, the roots of which will be the e quantities n, n', n'', ...

We now require to form equations for the roots in each period. This is shown to be possible by the following theorem.

§ 9. If $F([\lambda], [\lambda'], [\lambda''], \ldots)$ be any rational integral *symmetric* function of the roots of any period (f, λ), it may be expressed in the form $a + b(f, 1) + c(f, g) + \ldots + k(f, g^{e-1})$. (10)

1°. It is evident that F may be expressed in the form $A + Br + Cr^2 + \ldots + Kr^{p-1}$, or $A + B[1] + C[2] + \ldots + K[p-1]$, for each term of F must be the product of certain powers of r, and therefore itself a power of r, and its exponent may be made less than p since $r^p = 1$.

2°. The roots belonging to the same period will have equal coefficients and therefore may be summed under the form $M(f, \mu)$, say. Let $[\alpha]$, $[\beta]$ be a pair of roots belonging to a given period; we may suppose $\beta = \alpha h^m$, where $g^{(p-1)/f} = h$. In the identity

$$F([\lambda], [\lambda'], [\lambda''], \ldots) = A + B[1] + C[2] + \ldots + K[p-1]$$

substitute λh^m for λ. This will not alter the value of F, since it is a symmetric function of the roots of (f, λ), and hence we get

$$F([\lambda], [\lambda'], [\lambda''], \ldots) = A + B[h^m] + C[2h^m] + \ldots + K[(p-1)h^m].$$

Comparing these two expressions for F, we see that $[1]$ and $[h^m]$ have equal coefficients, and similarly with $[2]$ and $[2h^m]$, ... with $[\alpha]$ and

$[ah^m]$; *i.e.*, with any two roots of the same period. Hence the theorem follows.

Cor. Since the coefficients of an equation are symmetric functions of its roots, we see that the coefficients of an equation determining the roots of a given period may be expressed in the form (10).

§ 10. The two last paragraphs show us that if $p-1=ef$, we can make the solution of $x^p-1=0$ depend upon the solution of equations of the degrees e and f. The following theorem shows us that if f is not prime, we may make the solution depend on equations of still lower degree.

Let as in § 6, Cor. 2, $f=ab$ and F be a symmetric function of the periods (b, λ), $(b, \lambda h)$, ... then F may be expressed in the form

$$a+b(f, 1)+c(f, g)+ \ldots \tag{10}$$

By the last paragraph F may be put in the form $A+B(b, 1) + C(b, g)+ \ldots$

Now the periods (b, λ), $(b, \lambda h)$, ... of which (f, λ) is composed are unaltered when $\lambda h^{a.m}$ is put in place of λ, hence in $A+B(b, 1)+ C(b, g)+ \ldots$ there ought to be a term (b, a) which has the same coefficient as $(b, ah^{a.m})$ where m may have the values 1, 2, 3, ... $a-1$; *i e.*, all the periods forming (f, λ) have the same coefficient. F may therefore be put in the required form.

§ 11. We may now describe the general method of making the solution of $x^p-1=0$ (where p is a prime number) depend on equations of as low degree as possible.

1°. Find a primitive root g of the prime number p.

2°. Find the remainders (mod p) of the series 1, g, g^2, ... g^{p-2}.

3°. Resolve $p-1$ into its prime factors, say $p-1=abc \ldots k$.

4°. As in § 6, write the roots in a rectangular array, the first a in the first column, the second a in the second column, and so on, thus getting a periods of $bc \ldots k$ terms. Treat similarly the roots of each of these periods, getting b periods of $c \ldots k$ terms from each; and so on.

5°. Form an equation (A) (Cor. § 8), which has for its roots the a periods; any root of this may be taken as the value of $(bc \ldots k, 1)$, for *any* root of $X=0$ may be called r^1, and therefore any period may be considered as including [1]. The other periods $(bc \ldots k, g)$, $(bc \ldots k, g^2)$, ... may now be determined by § 8. Hence it is necessary to find only one root of (A).

We may distinguish the roots also by putting

$$[1] = \cos\frac{2m\pi}{p} + i\sin\frac{2m\pi}{p},$$

(where m is not divisible by p) and hence calculating from a table of sines and cosines the values of [2], [3],...with sufficient accuracy to determine their relative magnitude. We can thus distinguish the roots of (A) which should be denoted by (bc ... k, 1), (bc ... k, g), ... respectively.

6°. Now, form an equation (B) (§ 10), which has for its roots the b periods contained in (bc ... k, 1). As before, we may arbitrarily assign any root of (B) as the value of (c ... k, 1) and calculate the value of each similar period; or distinguish the roots by the help of a table of sines as above. Proceeding in this way we at last find the values of (k, 1), (k, g), ...

7°. Now, form an equation (K) (Cor. § 9), which has for its roots the roots of $X = 0$ contained in (k, 1). Any root of (K) being called [1], its successive powers will give all the other roots of $X = 0$.

§ 12. It is evident from the last paragragh that if $p - 1 = 2^a.3^b.5^c$... the solution of $X = 0$ may be made to depend on a equations of the 2nd degree, b equations of the 3rd degree, c of the 5th, &c.

That the solution may depend on quadratic equations only, we must have $p - 1 = 2^a$, *i.e.*, p must be of the form $2^a + 1$ and be a prime number. If a be odd then $2^a + 1$ is divisible by $2 + 1$ and hence p is not prime. Further if a contain an odd factor, say $a = bc$ where c is odd, then $2^{bc} + 1$ is divisible by $2^b + 1$ and again p is not prime. That the form $2^a + 1$ may be prime it is therefore *necessary* that a should be of the form 2^m, *i.e.*, $p = 2^{2^m} + 1$. But this condition is not *sufficient* (as stated by Fermat) for Euler has shown that $2^{32} + 1$ (4,294,967,297) is divisible by 641.

When $m = 0, 1, 2, 3, 4$; p becomes 3, 5, 17, 257, 65537 which are all prime. Hence the corresponding roots of unity may be found by quadratic equations only. Further, we may inscribe in a circle regular polygons of 3, 5, 17, &c., sides by means of ruler and compasses. The case of a 17-gon is given later on.

[From other considerations we know that if an m-gon and an n-gon (m prime to n) can be inscribed in a circle, so also may an mn-gon. Further, we may inscribe a polygon having double the number of sides of any inscribed polygon. Hence an n-gon may be inscribed in a circle if n contains no *odd* factor except of the form $2^{2^m} + 1$, each such factor being prime and not repeated.]

§ 13. As illustrative of the above, let us consider the case where $p=17$.

In the following table, each column (except the first) gives the numbers which are congruent (mod 17) to the successive powers of the first number in the column. The first column gives the corres-

1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2		4	9	16	8	2	15	13	13	15	2	8	16	9	4	1
3		8	10	13	6	12	3	2	15	14	5	11	4	7	9	
4		16	13	1	13	4	4	16	16	4	4	13	1	13	16	
5		15	5		14	7	11	9	8	6	10	3		12	2	
6		13	15		2	8	9	4	4	9	8	2		15	13	
7		9	11		10	14	12	15	2	5	3	7		6	8	
8		1	16		16	16	16	1	1	16	16	16		16	1	
9			14		12	11	10			7	6	5		3		
10			8		9	15	2			2	15	9		8		
11			7		11	5	14			3	12	6		10		
12			4		4	13	13			13	13	4		4		
13			12		3	10	6			11	7	14		5		
14			2		15	9	8			8	9	15		2		
15			6		7	3	5			12	14	10		11		
16			1		1	1	1			1	1	1		1		

ponding exponent. For example, the column headed 6 gives the remainders when 6, 6^2, 6^3,... respectively are divided by 17.

This readily exemplifies the theorems in § 1 and Corollaries.

The theorem of § 2 and Fermat's theorem are likewise illustrated; *e.g.*, the only exponents to which numbers *belong* in this table are 1, 2, 4, 8, and 16, all of which are factors of $17-1$.

§ 4 also finds illustration; *e.g.*, belonging to the exponent 8 are the numbers 2, 8, 9, 15; each of these numbers occurs as a remainder in the columns headed by these numbers; the corresponding exponents are always prime to 8 (unity being considered as prime to every number as in § 3); and the exponents corresponding to the other remainders are not prime to 8; lastly, 17 has a primitive root. (In fact there are $\phi 16 = 8$ primitive roots—viz., 3, 5, 6, 7, 10, 11, 12, and 14.)

§ 14. In general we need to find only one primitive root, and this may usually be done most simply by successive trial of the small numbers 2, 3,... Use should be made of the results in §§ 2 and 4. *E.g.* By trial we find the remainders of 2, 2^2, 2^3,... to be 2, 4, 8, 16, 15, 13, 9, 1. As a second trial we might take any of the numbers *not* contained in this series. In this case this is unnecessary, for, since 8 contains all the divisors of 16 (except 16 itself), we see that only the above numbers can belong to exponents less than 16, and hence the primitive roots of 17 are 3, 5, 6, 7, 10, 11, 12, 14, as above.

We may now arrange the roots in two periods as in § 6. We thus get:

(8, 1) containing [1], [9], [13], [15], [16], [8], [4], [2].
(8, 3) „ [3], [10], [5], [11], [14], [7], [12], [6].

Calling these periods n and n' respectively, we have:

$$n+n'=(16,\ 1)=-1 \quad (a);$$

and § 7,
$$\begin{aligned} nn' &= (8,4)+(8,11)+(8,6)+(8,12)+(8,15)+(8,8)+(8,13)+(8,7)\\ &= n + n' + n' + n' + n + n + n + n'\\ &= 4(n+n') = -4 \quad (b), \end{aligned}$$

and therefore n and n' are the roots of $n^2+n-4=0$.

We now break up the above periods into smaller periods, and get from the period (8, 1)

(4, 1) containing [1], [13], [16], [4] say m
(4, 9) „ [9], [15], [8], [2] „ m'.

Also from (8, 3) we get

(4, 3) containing [3], [5], [14], [12] say m''
(4, 10) „ [10], [11], [7], [6] „ m'''.

Here $\quad m+m'=(8,\ 1)=n$, say $\quad (c)$

and $$mm' = (4,\ 10) + (4,\ 16) + (4,\ 9) + (4,\ 3)$$
$$= m''' + m + m' + m''$$
$$= n + n' = -1 \ (d)$$

and hence (4, 1) and (4, 9) are the roots of $m^2 - nm - 1 = 0$. Similarly (4, 3) and (4, 10) are the roots of $m^2 - n'm - 1 = 0$, for we have $m'' + m''' = n'$ (e) and $m''m''' = -1$ (f).

[To illustrate the application of § 9 Cor., we may find an equation for the roots contained in the period (4, 1). Let $x^4 - Ax^3 + Bx^2 - Cx + D = 0$ be the required equation. $A = \Sigma x' = m$

$$\Sigma x'x'' = [14] + [17] + [5] + [12] + [17] + [3] = 2 + m'' = B$$
$$\Sigma x'x''x''' = [16] + [4] + [1] + [13] = m = C$$

and $x'x''x'''x'''' = 1$. Therefore the equation for the roots r^1, r^{13}, r^{16}, r^4 of the original equation is $x^4 - mx^3 + (2 + m'')x^2 - mx + 1 = 0$ (A) where m and m' are known. Any root of this equation may be called r, the others being determined by their relationship to r.

By symmetry we have other 3 equations to determine the other 12 roots; or all the roots may be determined from the value of r, *any* root of (A) by forming the powers $r^2, r^3, \ldots r^{16}$.]

Continuing, however, the process of separating the periods into lower periods we get:

(2, 1)	containing	[1], [16]	(2, 3)	containing	[3], [14]
(2, 13)	„	[13], [4]	(2, 5)	„	[5], [12]
(2, 9)	„	[9], [8]	(2, 10)	„	[10], [7]
(2, 15)	„	[15], [2]	(2, 11)	„	[11], [6]

Calling these periods $l_1, l_2 \ldots l_8$, we have

$$l_1 + l_2 = m \ (g), \text{ and } l_1 \, l_2 = l_5 + l_6 = m'' \ (h);$$

$\therefore$ $l^2 - ml + m'' = 0$(B) has (2, 1) and (2, 13) for roots.

Lastly $[1] + [16] = l_1$ and $[1].[16] = 1$ and therefore r is a root of $r^2 - l_1 r + 1$ (C).

It is easy to see that the reciprocal equation (A) is equivalent to the two quadratics (B) and (C).

§ 15. Inscription of a regular 17-gon in a given circle.

If AB be the side of a regular 34-gon then $AB = 2 \sin \frac{\pi}{34}$ (the radius being considered as unity), *i.e.* $AB = 2 \cos 4 \frac{2\pi}{17}$.

Now if we put $$[1] = \cos \frac{2\pi}{17} + i \sin \frac{2\pi}{17}$$

then $$[16] = \cos \frac{2\pi}{17} - i \sin \frac{2\pi}{17}$$

and $\qquad (2, 1)=2\cos\frac{2\pi}{17}$. Accordingly AB $=(2, 4)$.

From a table of cosines we get the following values to enable us to distinguish the roots:

$$l_1=1\cdot87\,;\ l_2=\cdot18\,;\ l_3=-1\cdot97\,;\ l_4=1\cdot48\,;$$
$$l_5=\cdot89\,;\ l_6=-\cdot55\,;\ l_7=-1\,70\,;\ l_8=-1\cdot21.$$

From these we get $m=2\cdot05$; $m'=-\cdot49$; $m''=\cdot34$; $m'''=-2\cdot91$. Lastly, $n=1\cdot56$ and $n'=-2\cdot56$.

CONSTRUCTION :—Draw OX (fig. 3) a tangent and OY a diameter to the given circle with centre A and radius unity. Consider through out lengths to the right of O as positive and to the left as negative. In OX take OB $=-\frac{1}{4}$. With B as centre and radius BA describe a circle meeting OX in C and D, then 2OC, 2OD are the roots of (a) and (b). For $2OC.2OD=-4OA^2=-4=nn'$ and $2OC+2OD=4OB=-1=n+n'$. By the above values we see that $2OC=n$ and $2OD=n'$.

With C as centre and CA as radius describe the circle EAF.

Then $\qquad OE.OF=-OA^2=-1=mm' \quad (d)$

and $\qquad OE+OF=2OC=n \quad (c)$

Hence, considering the above values, we get $OE=m$.

With D as centre and DA as radius describe the circle GAH.

Then $\qquad OG.OH=-OA^2=-1 \quad (f)$

and $\qquad OG+OH=2OD=n' \quad (e)$

Hence $\qquad OG=m''$

Make $OK=OA$ and on GK describe a semicircle meeting OY in L. Through M, the middle point of OE draw a parallel to OY and through L a parallel to OX. Let these meet in P. With P as centre and PL as radius describe a circle meeting OX in N and Q. Then $ON.OQ=OL^2=OG.OK=OG=m''$ (h); and $ON+OQ=2OM=OE=m$ (g). $ON=l_2=(2, 4)=$ the length of the side of a regular 34-gon.

The above construction is based on that given in Serret's *Algèbre Supérieure*. A geometrical analysis of the problem is given in Catalan's *Théorèmes et Problèmes de Géométrie Élémentaire*, 6ème éd. p. 267 (1879), and also (somewhat simplified) in the appendix to Casey's *Elements of Euclid*. Geometrical constructions are also given, by H. Schrœter, in *Crelle's Journal* (1872), translated in *Nouvelles Annales de Mathématiques* (1874), and by v. Staudt in *Crelle's Journal* (1842).

Fourth Meeting, February 8th, 1889.

GEORGE A. GIBSON, Esq., M.A., President, in the Chair.

On a Proposition in Statics.

By Professor C. NIVEN.

It is known that a force acting along any line in space may be resolved into six components. In the most commonly employed resolution these are forces along three lines at right angles, and couples round these lines. But the six components may be taken to be forces along the six edges of a tetrahedron. It is the object of what follows to determine these forces.

Let the given force be F, and let the line in which it acts be the intersection of the planes

$$\left.\begin{array}{l} Ax + By + Cz + Du = 0 \\ A'x + B'y + C'z + D'u = 0 \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

x, y, z, u being respectively the forces OBC, OCA, OAB, ABC, (fig. 4) of the tetrahedron OABC (in quadriplanar co-ordinates).

Let the forces in OA, OB, OC, BC, CA, AB be x, y, z, P, Q, R, and let the perpendiculars on BC, CA, AB and the line of action of F be p, q, r, f; let also equal and opposite forces $\pm$P, $\pm$Q, $\pm$R, $\pm$F be applied at O parallel to their former directions. The force F at O is equivalent to the forces x, y, z, P, Q, R at O, and the couple Ff is the resultant of the three couples Pp, Qq, Rr acting in the planes OBC, OCA, OAB.

The plane in which Ff acts is given, from (1), by drawing the plane through O and the line F; its equation is

$$(AD' - A'D)x + (BD' - B'D)y + (CD' - C'D)z = 0 \quad \ldots \quad \ldots \quad (2).$$

Let us draw a parallel plane A′B′C′ to this; its equation will be

$$(AD' - A'D)x + (BD' - B'D)y + (CD' - C'D)z = E \quad \ldots \quad \ldots \quad (3).$$

Now, couples in the faces of the tetrahedron, or in planes parallel to these faces, will balance each other if their moments are

proportional to the areas in which they act. This may be easily proved by going round each face in the right-handed direction, as viewed from the outside, and placing forces in each side proportional to it. All these forces mutually cancel, but those in any one face are equivalent to a couple proportional to the area of the force, thus, for example, the forces along the sides of the triangle A′B′C′ (fig. 5) make up a couple, whose moment is $2\mu.\Delta A'B'C'$.

Returning to our original figure (fig. 4), we find that

$$Pp : Ff = \Delta OB'C' : \Delta A'B'C' \quad \ldots \quad \ldots \quad \ldots \quad (4).$$

But the areas of the faces of a tetrahedron are inversely proportional to the perpendiculars on them from the opposite vertices.

To find these, let three lines at right angles be chosen, as axes of ξ. η ζ, and let the faces of the tetrahedron OABC referred to these axes be

$$\left.\begin{array}{l} x \equiv \xi\cos\alpha_1 + \eta\cos\beta_1 + \zeta\cos\gamma_1 = 0 \\ y \equiv \xi\cos\alpha_2 + \eta\cos\beta_2 + \zeta\cos\gamma_2 = 0 \\ z \equiv \xi\cos\alpha_3 + \eta\cos\beta_3 + \zeta\cos\gamma_3 = 0 \\ u \equiv \xi\cos\alpha_4 + \eta\cos\beta_4 + \zeta\cos\gamma_4 - \pi = 0 \end{array}\right\} \quad \ldots \quad \ldots \quad (5).$$

Equation (3) now takes the form

$[(AD' - A'D)\cos\alpha_1 + (BD' - B'D)\cos\alpha_2 + (CD' - C'D)\cos\alpha_3]\xi$ + two similar terms $= E$,

and the perpendicular from O on $A'B'C' = E/Q$

where $Q^2 = (AD' - A'D)^2 + (BD' - B'D)^2 + (CD' - C'D)^2$

$- 2(BD' - B'D)(CD' - C'D)\cos yz$ − two similar terms $\quad \ldots \quad (6).$

The perpendicular from A′ on OB′C′ is $E/(AD' - A'D)$, hence from (4) and the remark following it

$$Pp = \frac{AD' - A'D}{Q}. \; Ff \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7).$$

It remains to determine f the perpendicular from O on F. Let f_1 and f_2 be the perpendiculars on the two planes (1), the angle between these planes being θ.

From fig. 6 we have

$$f_1 = f\sin\phi, \quad f_2 = f\sin(\theta - \phi)$$

whence
$$f^2\sin^2\theta = f_1^2 + f_2^2 + 2f_1f_2\cos\theta$$

$$\cos\theta = \frac{AA' + BB' + CC' + DD' - (AB' + A'B)\cos xy - \ldots}{\Delta_1 \,.\, \Delta_2}$$

where
$$\Delta_1^2 = A^2 + B^2 + C^2 + D^2 - 2AB\cos xy - \ldots$$
$$\Delta_2^2 = A'^2 + \quad \ldots \quad - 2A'B'\cos xy - \ldots$$

whence
$$\Delta_1^2\,\Delta_2^2\sin^2\theta = \Delta^2$$

where $\Delta^2 = \Sigma\{T_{xy,xy}\sin^2 xy + 2T_{xy,xz}\cos xy\cos xz - 2T_{xy,zu}\cos xy\cos zu - 2T_{xy}\cos xy\}$

wherein
$$T_{xy,xy} = (AB' - A'B)^2$$
$$T_{xy,xz} = (AB' - A'B)(AC' - A'C)$$
$$T_{xy,zu} = (AC' - A'C)(BD' - B'D) + (AD' - A'D)(BC' - B'C)$$
$$T_{xy} = (AC' - A'C)(BC' - B'C) + (AD' - A'D)(BD' - B'D) \quad (8).$$

Now $f_1 = D\pi/\Delta_1$ and $f_2 = D'\pi/\Delta_2$

whence, on reduction, we find $f^2\Delta^2 = Q^2\pi^2$.

Substituting in (7), and remembering that $\sin xu = \pi/p$, we obtain finally

$$P = \frac{AD' - A'D}{\Delta}\sin xu . F \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

Δ having the form given in (8).

Kötters synthetic geometry of algebraic curves—Part II., involutions of the second and higher order.

[See Index.]

Amsler's Planimeter.

By Professor STEGGALL.

There are many proofs of the principle of this planimeter, but all that are accessible to me seem a little beyond the grasp of many students who use the instrument. It seems worth while, therefore, to notice the following proof, which, to the best of my knowledge, is new.

Let A (fig. 7) be a pivot, round which the pivoted rods AB, BC rotate, and let AB′, B′C′ be a consecutive position of the rods, when the point C has traced out the arc CC′ of the curve whose area is required; and let us first suppose that this curve does not include the point A. The element of area ABCC′B′ consists of three parts, namely :—

(1) The triangle ABB′.

(2) The parallelogram BCC″B′, where B′C″ is equal and parallel to BC.

(3) The triangle C′B′C″.

Now, since the planimeter in its circuit returns to its original position, the areas (1) have a sum zero; so have the areas (3); and the area of the closed figure is, therefore, equal to the sum of all such parallelograms as BCC″B′; and this sum equals their common side BC, multiplied by their total heights.

Next, let P be any fixed point on BC, then its (elementary) motion at right angles to BC consists of the line P′M + MN, which is perpendicular to BC and B′C′; the sum of these elements is the registration by a wheel at P, free to revolve on BC as an axis; but the sum of the elements P′M vanishes (as is the case with the areas (1) and (3)) from its twofold description, and, therefore, the registration of the wheel is equal to the sum of all the heights of the parallelograms BCC′B″. Hence the area of the closed curve is equal to BC × registration of wheel, where "registration of wheel" means number of revolutions × circumference.

If the curve enclose the point A, as the instrument is not constructed to allow BC to cross over AB, we must note that our curve involves, besides the parallelograms, the areas of the circles whose radii are AB, BC; and the registration only gives us the heights of the parallelograms, together with the circumference of the circle of radius BP. Thus we must add to our result obtained by the usual reading, the expression $\pi AB^2 + \pi BC^2 - 2\pi BP.BC$. If a circle of known area is described about A, this quantity can be at once found for any particular instrument.

Fifth Meeting, March 8th, 1889.

George A. Gibson, Esq., M.A., President, in the Chair.

An Elementary Discussion of the Closeness of the Approximation in Stirling's Theorem.

By Prof. Chrystal.

[The substance of this paper will appear in the second volume of Professor Chrystal's Algebra.]

On the most Economical Speed to drive a Steamer in relation to the cargo carried and coals consumed.

By W. J. MILLAR, C.E.

The object of this communication is simply to show how a formula may be obtained which will indicate the most economical speed for a steamer in relation to cargo carried and coals consumed on the voyage.

Let V = speed of ship, say in knots.

„ C = total carrying capacity of ship (cargo and coal), say in tons.

„ T = time occupied on voyage, say in hours.

„ S = space traversed, or voyage, say in knots.

Let the power required to drive the ship be supposed to vary as V^3*, then

$$P = mV^3 \quad \ldots \quad \ldots \quad \ldots \quad (1).$$

The *work* done in traversing a given distance will vary as $P \times$ space, hence we may write for total work done on voyage

$$mV^3S. \quad \ldots \quad \ldots \quad \ldots \quad (2).$$

Since, however, this work is obtained from the energy contained in the coal carried, we may express this coal consumpt in tons as

$$K(mV^3S) \quad \ldots \quad \ldots \quad \ldots \quad (3).$$

Hence for the clear cargo capacity we have

$$C - K(mV^3S) \quad \ldots \quad \ldots \quad \ldots \quad (4).$$

Now let the marketable value of the cargo be assumed to vary inversely as the time taken on the voyage, then we may write, as representing this value, $\{C - K(mV^3S)\}/T \quad \ldots \quad \ldots \quad (5).$

But $T = S/V$ hence we have by substitution

$$(CV/S) - K(mV^4) \quad \ldots \quad \ldots \quad (6).$$

Now, since, by the question, this must be a maximum, differentiate and we have

$$(CdV/S) - 4KmV^3dV = 0$$

or

$$C/4KmS = V^3$$

hence

$$V = \sqrt[3]{C/4KmS} \quad \ldots \quad \ldots \quad (7).$$

The most economical speed, therefore, at which the vessel should be driven to fulfil the conditions is that which is equal to the cube root of the total carrying capacity of the ship divided by the product of the space travelled, or voyage completed, and the constants noted.

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* Comparisons of power and speed at different speeds show that the exponent of speed is not, however, a constant quantity, but varies itself to some extent with the speed.

To give an example of application of the rule obtained :—

An Atlantic steamer propelled at an average speed of 16 knots per hour by engines indicating 6000 horse power, with an expenditure of, say, 1000 tons of coal on a voyage of 3000 knots.

The cargo carrying capacity of the vessel may be taken at 3000 tons.

First, to find the value of the constant m, we have $6000 = m16^3$ (see (1)),

hence $$m = 1\tfrac{1}{2} \text{ nearly.}$$

Second, to find the value of the constant K we have

$$1000 = K \times 1\tfrac{1}{2} \times 16^3 \times 3000 \quad \ldots \quad \ldots \quad (3),$$

hence $$K = \text{say } \frac{1}{18{,}000}.$$

The value of V (see (7)) will therefore be expressed as

$$V = \sqrt[3]{3000 \times C/S}.$$

Now to apply this rule to the case of the same ship or another of same size and power, making a longer voyage, say of 12,000 miles, we have

$$V = \sqrt[3]{(3000 \times 4000/12{,}000)} = 10 \text{ knots},$$

hence the most economical speed for a voyage of this latter extent becomes 10 knots, as compared with 16 knots for a voyage of 3000 knots.

Note.—It has been assumed, for purposes of simplicity, in the "Paper," that the commercial value of the cargo varies inversely as the time occupied on the voyage.

Clearly this is an assumption which might seldom occur in practice, there being so many conflicting variable elements in connection with such a question viewed purely from a commercial standpoint.

Doubtless, however, there are cases where the assumption cannot be so far from what occurs in practice—*e.g.*, the case of the China tea clipper, or perishable cargoes, such as fruit.

The investigation is simply made with the view of ascertaining the most economical relation between speed, cargo carried, and coal consumed, where certain data are given.

The author has looked at the question more from a mathematical than from a commercial point of view, as it seemed to indicate an investigation where mathematics might be used to point to certain conclusions, which, although not always obtained in practice, yet

might be approximated to; the mathematical or physical treatment pointing out the way in which the subject might be treated.

Kötter's Synthetic Geometry of Algebraic Curves—Part III., Involution Nets, and Involutions of 2nd, 3rd, ... Rank.

By Rev. NORMAN FRASER, B.D.

[See Index.]

Sixth Meeting, 12*th April* 1889.

A. Y. FRASER, Esq., M.A., Vice-President, in the Chair.

On Vortex Motion in a rotating fluid.

By C. CHREE, M.A.

The object of the following paper is to consider the motion of one or more vortices in a compressible fluid, which is rotating as a whole with uniform angular velocity ω about an axis, taken as axis of z. To save space I shall when possible refer for results to a previous paper in the *Proceedings*, distinguishing the equations of that paper, Vol. V., pp. 52-59, by the suffix a.

Formulæ for fluid motion relative to rotating axes are given by Greenhill in the article "Hydrodynamics" in the *Encyclopaedia Britannica*, and by Basset in his "Treatise on Hydrodynamics," Vol. I., § 23. The velocities appearing however in these equations are partly at least absolute velocities, while equations containing only velocities relative to the moving axes seem most suitable for our purpose. Such equations may be obtained shortly as follows, confining our attention to the case when there is no velocity parallel to the axis of rotation.

Let u, v denote the velocity components relative to the moving axes ox, oy in the fluid at the point x, y at the time t, and let u', v'

denote the velocity components relative to fixed axes with which the moving axes coincide at the instant considered. Then we have

$$u' = u - \omega y,\ v' = v + \omega x \quad \ldots \quad \ldots \quad \ldots \quad (1);$$

whence

$$\left.\begin{aligned} \frac{du'}{dx} &= \frac{du}{dx}, \quad \frac{dv'}{dx} = \frac{dv}{dx} + \omega \\ \frac{du'}{dy} &= \frac{du}{dy} - \omega, \quad \frac{dv'}{dy} = \frac{dv}{dy} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (2).$$

At the end of an indefinitely short interval τ the volocity components at the point x, y relative to the moving axes are

$$u + \frac{du}{dt}\tau, \text{ and } v + \frac{dv}{dt}\tau \quad \ldots \quad \ldots \quad \ldots \quad (3);$$

where $\frac{d}{dt}$ denotes partial differentiation and is employed when there is no variation in the co-ordinates x, y. At the end of this interval the axes of x and y make angles $\omega\tau$ with the fixed axes ox', oy' with which they coincided at the time t. Thus the co-ordinates of the point, which we denote above by x, y, referred to the fixed axes are

$$\left.\begin{aligned} x' &= x\cos\omega\tau - y\sin\omega\tau = x - \omega\tau y \text{ in limit} \\ y' &= y\cos\omega\tau + x\sin\omega\tau = y + \omega\tau x \text{ in limit} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (4).$$

Thus the velocity component in the direction ox' at the above point is

$$u' + \frac{du'}{dt}\tau + \frac{du'}{dx}(x' - x) + \frac{du'}{dy}(y' - y) \quad \ldots \quad \ldots \quad (5);$$

where $\frac{du'}{dt}$ denotes partial differentiation in the sense that is usual in the equations of motion, and signifies the variation that occurs in the velocity at a point absolutely fixed in space. In the limit when τ is very small (5) transforms into

$$u' + \tau\left(\frac{du'}{dt} - \omega y\frac{du'}{dx} + \omega x\frac{du'}{dy}\right);$$

and from (1) and (2) this is identical with

$$u - \omega y + \tau\left(\frac{du'}{dt} - \omega y\frac{du}{dx} + \omega x\frac{du}{dy} - \omega^2 x\right) \quad \ldots \quad \ldots \quad (5').$$

But again from (1) we have for the velocity relative to the fixed axis ox' at time $t + \tau$ the expression

$$\left(u + \frac{du}{dt}\tau - \omega y\right)\cos\omega\tau - \left(v + \frac{dv}{dt}\tau + \omega x\right)\sin\omega\tau;$$

which becomes in the limit

$$u - \omega y + \tau\left(\frac{du}{dt} - \omega v - \omega^2 x\right) \quad \ldots \quad \ldots \quad (6).$$

Equating (5′) and (6), we find

$$\frac{du'}{dt}=\frac{du}{dt}-\omega v+\omega y\frac{du}{dx}-\omega x\frac{du}{dy}\quad \dots \quad \dots \quad (7).$$

Again, from (1) and (2) we have

$$u'\frac{du'}{dx}+v'\frac{du'}{dy}=(u-\omega y)\frac{du}{dx}+(v+\omega x)\left(\frac{du}{dy}-\omega\right)\quad \dots \quad (8).$$

If we use p, ρ, X, Y in their usual sense, the equations of motion referred to the fixed axes ox', oy' are

$$\frac{du'}{dt}+u'\frac{du'}{dx}+v'\frac{du'}{dy}=\mathrm{X}-\frac{1}{\rho}\frac{dp}{dx}\quad \dots \quad \dots \quad (9),$$

$$\frac{dv'}{dt}+u'\frac{dv'}{dx}+v'\frac{dv'}{dy}=\mathrm{Y}-\frac{1}{\rho}\frac{dp}{dy}\quad \dots \quad \dots \quad (10).$$

Now, adding (7) and (8) we transform (9) into

$$\frac{\delta u}{\delta t}-2\omega v-\omega^2 x=\mathrm{X}-\frac{1}{\rho}\frac{dp}{dx}\quad \dots \quad \dots \quad \dots \quad (11),$$

where
$$\frac{\delta u}{\delta t}\equiv\left(\frac{d}{dt}+u\frac{d}{dx}+v\frac{d}{dy}\right)u\quad \dots \quad \dots \quad (12).$$

Remembering the meaning that now attaches to $\frac{du}{dt}$ we see that the operator $\frac{\delta}{\delta t}$ denotes differentiation following the fluid, and so has exactly the same meaning as in my former papers.

Similarly (10) transforms into

$$\frac{\delta v}{\delta t}+2\omega u-\omega^2 y=\mathrm{Y}-\frac{1}{\rho}\frac{dp}{dy}\quad \dots \quad \dots \quad \dots \quad (13).$$

If it be assumed that p is a function of ρ and that X, Y are derivable from a potential, or vanish, we get by differentiating (13) with respect to x and subtracting (11) differentiated with respect to y

$$\frac{\delta\zeta}{\delta t}+(\zeta+\omega)\left(\frac{du}{dx}+\frac{dv}{dy}\right)=0\quad \dots \quad \dots \quad (14),$$

where $2\zeta\equiv\frac{dv}{dx}-\frac{du}{dy}$.

We next must find the form of the equation of continuity containing u and v. Let us suppose dx, dy to form adjacent sides of the section of a rectangular prism, infinite in the direction of z, which is fixed in space so that its sides dx, dy coincide in direction with the instantaneous positions of the axes ox, oy at time t. The fluid velocities normal to the faces dx, dy are respectively $v+\omega x$ and $u-\omega y$. Thus if $\frac{d'\rho}{dt}$ indicates that the point where the variation in

density is being measured is absolutely fixed in space, the ordinary equation of continuity for motion in two dimensions is

$$\frac{d.}{dx}\rho(u-\omega y)+\frac{d.}{dy}\rho(v+\omega x)+\frac{d'\rho}{dt}=0 \quad \ldots \quad \ldots \quad (15).$$

But if $\frac{d\rho}{dt}$ indicates that the point where the density is being measured is fixed relative to the moving axes, we get

$$\frac{d\rho}{dt}\tau=\frac{d'\rho}{dt}\tau+(x'-x)\frac{d\rho}{dx}+(y'-y)\frac{d\rho}{dy},$$

where x', y' are given by (4). Proceeding to the limit, we get

$$\frac{d'\rho}{dt}=\frac{d\rho}{dt}+\omega y\frac{d\rho}{dx}-\omega x\frac{d\rho}{dy}.$$

Thus (15) simply transforms into

$$\frac{d(\rho u)}{dx}+\frac{d(\rho v)}{dy}+\frac{d\rho}{dt}=0 \quad \ldots \quad \ldots \quad \ldots \quad (15').$$

This is identical in *form* with the ordinary equation when the axes are fixed; but the difference in the meaning of the symbols, particularly of $\frac{d\rho}{dt}$, must be carefully attended to.

Using $\frac{\delta}{\delta t}$ in the sense of equation (12), we can transform (15') into

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{1}{\rho}\frac{\delta\rho}{\delta t}=0 \quad \ldots \quad \ldots \quad \ldots \quad (15'').$$

Combining (14) and (15'') we get

$$\frac{\delta}{\delta t}\left(\frac{\zeta+\omega}{\rho}\right)=0 \quad \ldots \quad \ldots \quad \ldots \quad (16).$$

Thus the apparent vorticity ζ of an element remains constant if the fluid be incompressible, but in a gas the apparent vorticity will vary unless the density remain constant. If we suppose the density to be the same over the cross section σ of the vortex, then since $\sigma\rho$ is necessarily constant in virtue of the principle of continuity, we may replace (16) by

$$\sigma(\zeta+\omega)=m'(a \text{ constant}) \quad \ldots \quad \ldots \quad \ldots \quad (17).$$

If $m\equiv\sigma\zeta$ denote the apparent strength of the vortex, then it follows that

$$m=m'-\omega\sigma \quad \ldots \quad \ldots \quad \ldots \quad (18).$$

Thus while the absolute strength m' is constant, the apparent strength m will vary unless the cross section, and so the density, remains constant. The relation between the rates of variation of the apparent strength and the cross section, or the density, is by (18)

$$\frac{\delta m}{\delta t}=-\omega\frac{\delta\sigma}{\delta t}=\frac{\omega\sigma}{\rho}\frac{\delta\rho}{\delta t} \quad \ldots \quad \ldots \quad \ldots \quad (19).$$

For brevity we may term a vortex a *cyclone* or an *anticyclone* according as its vorticity is of the same or opposite sign to ω. We see from (19) that the apparent strength of a cyclone increases or diminishes according as its density is increasing or diminishing. The reverse is the case with the numerical value of the strength of an anticyclone.

From (19) it follows as a particular case than an element of fluid originally devoid of apparent vorticity cannot alter in density without developing vorticity. The vorticity developed will be cyclonic or anticyclonic according as the density increases or diminishes. It must be borne in mind that the density here referred to is that of a particular element of the fluid, and not the density found at a point fixed relative to the rotating axes.

The remarks made in my previous papers on the case of a fluid bounded by an infinite plane, and acted on by forces at right angles to, and functions only of the distance from, that plane, apply equally to the case of a rotating fluid, provided the infinite plane be perpendicular to the axis of rotation. The remarks on the applicability of our formulæ, in cases where ρ and ζ vary over the cross section of a vortex, require no modification.

From (15″) we see that the formulæ for the velocity components relative to the moving axes, due to the variation of density in the rotating fluid, must be the same as those for the velocity components in the ordinary case of fixed axes and a non-rotating fluid. Thus the velocity components in the case of a rotating fluid and axes are given by the formulæ (4_a) and (5_a), viz.:—

$$u = -\frac{my}{\pi r^2} + \frac{1}{2\pi}\frac{\delta\sigma}{\delta t}\frac{x}{r^2}, \qquad v = \frac{mx}{\pi r^2} + \frac{1}{2\pi}\frac{\delta\sigma}{\delta t}\frac{y}{r^2},$$

remembering that m is no longer a constant, but equal to $m' - \omega\sigma$ where m' is a constant.

If ϕ be an angle measured from the plane of xz, the stream lines due to the vortex are given by the differential equation

$$\frac{1}{r^2}\frac{dr^2}{dt} = \frac{1}{m' - \omega\sigma}\frac{\delta\sigma}{\delta t} \times \frac{d}{dt}(\tan^{-1}y/x) \quad \ldots \quad \ldots \quad (20).$$

This admits of immediate integration if

$$\frac{1}{m' - \omega\sigma}\frac{\delta\sigma}{\delta t} = \nu, \text{ (a constant)} \quad \ldots \quad \ldots \quad (21);$$

or

$$m' - \omega\sigma = (m' - \omega_0\sigma)e^{-\omega\nu}t \ \ldots \quad \ldots \quad \ldots \quad (22),$$

where ${}_0\sigma$ is the value of σ when $t = 0$.

In this case the integral of (20) is

$$r = ae^{\nu\phi/2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (23)$$

where a is the value of r when $\phi = 0$.

When ν is very small we observe that so long as t is not very large, (22) is approximately of the form

$$\sigma = {}_0\sigma + \sigma' t + \sigma'' t^2 + \quad \ldots \quad \ldots \quad \ldots \qquad (24),$$

where σ', σ'', &c., are constants of which σ'' is small compared to σ'. Thus when considering what happens within a comparatively short interval of a given instant, we might in such a case neglect all but the first two terms.

For the action on each other of two vortices, distinguished by the suffixes 1 and 2, we have equations $(7_a) - (10_a)$, with m representing apparent vorticity. Whence for the distance of the vortices at time t, and the inclination of the line joining their centres to the axis of x, we get

$$r^2 = a^2 + \frac{1}{\pi}(\sigma_1 + \sigma_2 - {}_0\sigma_1 - {}_0\sigma_2) \quad \ldots \quad \ldots \quad \ldots \qquad (25),$$

$$\phi = \frac{1}{\pi}\int_0^t \frac{m_1' + m_2' - \omega(\sigma_1 + \sigma_2)}{a^2 + \frac{1}{\pi}(\sigma_1 + \sigma_2 - {}_0\sigma_1 - {}_0\sigma_2)}\,dt \quad \ldots \qquad (26).$$

If the variations of the cross sections be given by equations of the type (24) with $\sigma'' = 0$, we find

$$r^2 = a^2 + \frac{1}{\pi}(\sigma_1' + \sigma_2')t \quad \ldots \quad \ldots \quad \ldots \qquad (27),$$

$$\phi = \frac{m_1' + m_2' - \omega({}_0\sigma_1 + {}_0\sigma_2) + \omega\pi a^2}{\sigma_1' + \sigma_2'} \log\left\{1 + \frac{(\sigma_1' + \sigma_2')t}{\pi a^2}\right\} - \omega t \qquad (28),$$

$$= \frac{{}_0m_1 + {}_0m_2 + \omega\pi a^2}{\sigma_1' + \sigma_2'} \log\frac{r^2}{a^2} - \frac{\omega\pi(r^2 - a^2)}{\sigma_1' + \sigma_2'} \quad \ldots \quad \ldots \qquad (28'),$$

where as usual the suffix 0 denotes values at the instant $t = 0$.

If we suppose $(\sigma_1' + \sigma_2')/\pi a^2$ to be very small, then, so long as t is not very large, we have: as a first approximation

$$\phi = ({}_0m_1 + {}_0m_2)t/\pi a^2 = \frac{{}_0m_1 + {}_0m_2}{\sigma_1' + \sigma_2'}\left(\frac{r^2}{a^2} - 1\right) \quad \ldots \quad \ldots \qquad (28''),$$

and as a second approximation

$$\phi = ({}_0m_1 + {}_0m_2)t/\pi a^2 - ({}_0m_1 + {}_0m_2 + \omega\pi a^2)(\sigma_1' + \sigma_2')t^2/2\pi^2 a^4 \qquad (28''').$$

So long as terms in t^3 are neglected, the result obtained from the more general form (24), retaining σ'', would not differ from (28''').

In cases where (28‴) applies, we see that the angular velocity of the line joining the vortices increases or diminishes as t increases, according as $(\sigma_1' + \sigma_2')\{1 + \omega\pi a^2/({}_0m_1 + {}_0m_2)\}$ is negative or positive. If ${}_0m_1 + {}_0m_2$ have the same sign as ω, the angular velocity thus increases when the sum of the cross sections diminishes, and conversely. The same law holds good when ${}_0m_1 + {}_0m_2$ is of opposite sign to ω and numerically greater than $\omega\pi a^2$; but it must be reversed when ${}_0m_1 + {}_0m_2$ is numerically less than $\omega\pi a^2$.

Returning to the general equations $(7_a) - (10_a)$, and putting

$$\left.\begin{aligned} m_1'x_1 + m_2'x_2 &= (m_1' + m_2')\mathrm{X} \\ m_1'y_1 + m_2'y_2 &= (m_1' + m_2')\mathrm{Y} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (29),$$

we find, precisely, as we found (19a),

$$(m_1' + m_2')\frac{\delta \mathrm{X}}{\delta t} = \omega(m_2'\sigma_1 - m_1'\sigma_2)\frac{y_2 - y_1}{\pi r^2} + \frac{x_2 - x_1}{2\pi r^2}\left(m_2'\frac{\delta\sigma_1}{\delta t} - m_1'\frac{\delta\sigma_2}{\delta t}\right) \ldots (30),$$

$$(m_1' + m_2')\frac{\delta \mathrm{Y}}{\delta t} = -\omega(m_2'\sigma_1 - m_1'\sigma_2)\frac{x_2 - x_1}{\pi r^2} + \frac{y_2 - y_1}{2\pi r^2}\left(m_2'\frac{\delta\sigma_1}{\delta t} - m_1'\frac{\delta\sigma_2}{\delta t}\right) \ (31).$$

Whether σ_1 and σ_2 vary or not, it follows that X and Y are constants provided $m_2'/\sigma_2 = m_1'/\sigma_1$.

If ζ_1, ζ_2 be the apparent mean vorticities, this leads to $\zeta_2 = \zeta_1$. Thus two vortices which have at every instant the same mean vorticity or "concentration," and whose vorticities are in the same direction—whether coinciding or not with that of ω—have a fixed point answering to a centre of gravity about which they describe similar orbits. The distances of the vortices from this centre are inversely as their strengths real or apparent. This result, combined with the values already found for r and ϕ, determines in this case the motion of the vortices. If ω and the rates of variation of σ_1 and σ_2 be small, X and Y remain nearly constant, and approximate values for them can be easily deduced from (30) and (31). A general solution of these equations seems scarcely likely to be obtainable.

The case where the apparent strengths of the vortices are equal but opposite in sign presents certain peculiarities.

In every case we have

$$m_1 + m_2 = m_1' + m_2' - \omega(\sigma_1 + \sigma_2).$$

Now m_1', m_2' are constant, and so when $m_1 + m_2$ is zero we must have $\qquad \sigma_1 + \sigma_2 = \text{constant} \ldots \quad \ldots \quad \ldots \quad (32).$

Supposing, initially, $x_2 - b = -(x_1 - b) = a/2$, $y_2 = y_1 = 0$, where a and b are constants, we get in this case

$$\left.\begin{aligned} x_2 - b &= \frac{1}{2}a + \frac{1}{2\pi a}(\sigma_1 - {}_0\sigma_1) \\ x_1 - b &= -\frac{1}{2}a + \frac{1}{2\pi a}(\sigma_1 - {}_0\sigma_1) \\ y_2 = y_1 &= \frac{1}{\pi a}m_1' t - \frac{\omega}{\pi a}\int_0^t \sigma_1 dt \end{aligned}\right\} \quad \ldots \quad \ldots \quad (33).$$

The vortices thus remain a constant distance apart, and the line joining them retains a fixed direction relative to the rotating axes.

If the fluid be incompressible, or more generally if the density remain constant, this will solve the case of a vortex in presence of an infinite plane boundary parallel to the axis of rotation and rotating with the fluid. For the velocity is everywhere tangential to the plane $x = b$, and this may accordingly be supposed to become a fixed boundary.

If, however, the density of the vortex vary, the velocity in the case above would not be wholly tangential to the plane $x = b$, unless we had $\dfrac{\delta\sigma_1}{\delta t} = \dfrac{\delta\sigma_2}{\delta t}$.

This is obviously reconcileable with (32) only when σ_1 and σ_2 both remain constant.

The case of a vortex of varying density in a rotating fluid, in presence of an infinite wall, might thus, at first sight, seem insoluble by the method of images. For if we suppose m_1 to be the real vortex, its image m_2 would have to satisfy the condition $\dfrac{\delta\sigma_2}{\delta t} = \dfrac{\delta\sigma_1}{\delta t}$ to ensure the velocity being tangential to the wall, and simultaneously the condition $\dfrac{\delta\sigma_2}{\delta t} = -\dfrac{\delta\sigma_1}{\delta t}$ in order to retain a constant absolute strength. This last condition, however, though necessary in a vortex of real fluid, is in no way necessary in the case of a vortex image which has no material existence, and is merely a mathematical device for obtaining velocities which at every instant satisfy the equations in the interior and at the surface of a fluid. When the image does not satisfy all the conditions that a true material existence would demand, it might be as well to term it an *instantaneous* image, as merely satisfying the mathematical conditions at the instant considered.

The instantaneous image of a vortex of section σ and apparent strength m in presence of an infinite plane $x=b$, has a section σ and apparent strength $-m$; and the velocity of the vortex at any instant is thus given by

$$\frac{dx}{dt}=-\frac{1}{2\pi}\frac{\delta\sigma}{\delta t}\frac{1}{2(b-x)},$$

$$\frac{dy}{dt}=\frac{m}{\pi}\frac{1}{2(b-x)}.$$

Supposing $x=b-a$ and $\sigma={}_0\sigma$ when $t=0$, we find

$$\left.\begin{aligned} x &= b-\sqrt{a^2+\frac{1}{2\pi}(\sigma-{}_0\sigma)} \\ y &= \frac{1}{2\pi}\int_0^t\frac{(m'-\omega\sigma)dt}{\sqrt{a^2+\frac{1}{2\pi}(\sigma-{}_0\sigma)}} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (34).$$

The motion of the vortex is thus completely determined if the law of variation of σ be known.

The case of a straight vortex in a fluid bounded by a right circular cylinder, whose axis is parallel to the vortex, and coincides with the axis of rotation if the fluid be rotating, can also be solved by the method of images. Let the radius of the cylinder be c, and at time t let m and σ denote the apparent strength and cross section of the vortex, and r and ϕ the polar co-ordinates of the vortex centre referred to the axis of rotation, and an initial plane moving with the fluid. It is easily proved that the instantaneous image consists of a vortex, whose apparent strength is $-m$ and rate of variation of cross section $\frac{\delta\sigma}{\delta t}$, situated at the point $(c^2/r,\ \phi)$, and of a vortex of apparent strength 0 and rate of variation of cross section $-\frac{\delta\sigma}{\delta t}$ coinciding with the axis of rotation.

We thus get for the motion of the vortex the two equations

$$\frac{\delta r}{\delta t}=-\frac{1}{2\pi}\frac{\delta\sigma}{\delta t}\left(\frac{1}{r}+\frac{r}{c^2-r^2}\right) \quad \ldots \quad \ldots \quad \ldots \quad (35),$$

$$r\frac{\delta\phi}{\delta t}=\frac{mr}{\pi(c^2-r^2)} \quad \ldots \quad \ldots \quad \ldots \quad (36).$$

Supposing that when $t=0$, $r=r_0$ and $\sigma={}_0\sigma$, we find

$$r^2=c^2-\sqrt{(c^2-r_0{}^2)^2+2c^2(\sigma-{}_0\sigma)/\pi} \quad \ldots \quad \ldots \quad (37),$$

$$\phi=\int_0^t\frac{mdt}{\sqrt{\pi^2(c^2-r_0{}^2)^2+2\pi c^2(\sigma-{}_0\sigma)}} \quad \ldots \quad \ldots \quad (38).$$

When c/r_0 is very large, approximate values are

$$\left.\begin{aligned} r^2 &= r_0^{\,2} - (\sigma - {}_0\sigma)/\pi \\ \frac{\delta\phi}{\delta t} &= \frac{m}{\pi c^2}\left(1 + \frac{r^2}{c^2}\right) \end{aligned}\right\} \quad \dots \quad \dots \quad \dots \quad (39).$$

The vortex will thus approach to or recede from the axis of rotation according as its density is diminishing or increasing; and it will revolve about the axis of rotation in the same direction as the moving axes or in the opposite, according as it is a cyclone or an anticyclone. This rotation, it will be observed, takes place relative to the moving axes, and is in addition to the common angular velocity ω; for a vortex of given strength it will be greater the greater the distance of the vortex from the axis of rotation.

The question next arises as to how the density of the vortex compares with the density which would be found in the fluid at the same distance from the axis of rotation in the absence of all apparent vorticity.

Referring to (11) and (13) we see that when the fluid is undisturbed by the existence of vortices, or of external forces causing motion relative to the moving axes, we have, since $u = 0 = v$,

$$\frac{1}{\rho'}\frac{dp}{dx} = \omega^2 x, \quad \frac{1}{\rho'}\frac{dp}{dy} = \omega^2 y,$$

where ρ' denotes the density in the undisturbed state.

Supposing a constant force $Z = -g$ parallel to the axis of rotation, we also have

$$\frac{1}{\rho'}\frac{dp}{dz} = -g.$$

If the relation between the pressure and density be $p = k\rho$ where k is constant, we thence obtain at a distance r from the axis of rotation

$$\log(\rho'/\rho'_0) = -gz/k + \tfrac{1}{2}\omega^2 r^2/k \quad \dots \quad \dots \quad \dots \quad (40),$$

where ρ_0' is the density where the axis of rotation cuts the plane of xy. The density thus increases with the distance from the axis of rotation, and to a body moving along a radius vector the relation between the rate of variation in the density of the surrounding fluid and the velocity of the body is given by

$$\frac{1}{\rho'}\frac{\delta\rho'}{\delta t} = \frac{\omega^2}{k} r \frac{\delta r}{\delta t} \quad \dots \quad \dots \quad \dots \quad \dots \quad (41).$$

Referring to (35) we see that the rate of change of density in a single vortex in a cylinder of radius c and its radial velocity are connected by the relation

$$\frac{1}{\rho}\frac{\delta\rho}{\delta t} = \frac{2\pi}{\sigma}(1 - r^2/c^2) r \frac{\delta r}{\delta t} \quad \dots \quad \dots \quad \dots \quad (42).$$

In any given case, by comparing (41) and (42), we get the information desired.

Suppose, for example, the fluid atmospheric air, ω the earth's angular velocity, and r/c small. Then, if the imaginary body of equation (41) and the vortex be at the same distance from the axis, and possessed of the same velocity, we find approximately

$$\frac{\delta\rho}{\delta t} \Big/ \frac{\delta\rho'}{\delta t} = \frac{2k\rho}{\omega^2 e^2 \rho'},$$

where e is the radius of the vortex, supposed circular.

If R denote the earth's radius, and H the height of the homogeneous atmosphere, this leads to

$$\frac{\delta\rho}{\delta t} \Big/ \frac{\delta\rho'}{\delta t} = \frac{578\text{RH}}{e^2} \frac{\rho}{\rho'}, \text{ approximately.}$$

Thus $\frac{\delta\rho}{\delta t} > \text{or} < \frac{\delta\rho'}{\delta t}$, according as $e < \text{or} > \sqrt{578\text{RH}\rho/\rho'}$.

The critical value of e would be over 3000 miles unless the difference between ρ and ρ' exceeded the differences occuring in the earth's atmosphere.

Our formulæ would cease to be at all accurate if e/c ceased to be small, or if e became comparable with the distance of the vortex from the cylindrical boundary. So in the case considered, when our formulae are applicable $\frac{\delta\rho}{\delta t}$ would be certainly greater than $\frac{\delta\rho'}{\delta t}$ unless c were very much greater than R.

In the case of the earth's atmosphere there would appear a fair probability that the motion of a vortex in comparatively high latitudes would resemble, in its general features, the motion we have found for a vortex in a cylinder. The radius of the cylinder would be of the same order of magnitude as an earth's quadrant. The two main reasons for this statement are, 1°, that as the earth's atmosphere is presumably not being thrown off, the motion at every point on its limiting surface must be very approximately at right angles to the perpendicular on the axis of rotation, and, 2°, that the winds in the northern and southern hemispheres are on the whole independent of one another. The conclusions so deduced should certainly answer better than those derived from the hypothesis of a fluid extending to infinity in all directions at right angles to the axis of rotation. In this latter case we should have finite radial

velocities at all distances from the axis of rotation, and also at great distances from this axis an atmosphere of enormous density.

The conclusions derived from the motion of a single vortex in a cylinder would lead to the following laws for the motion of a solitary cyclone or anti-cyclone in the earth's northern hemisphere in high latitudes.

1°. Motion in longitude:—

(α). The vortex moves from west to east if a cyclone, and from east to west if an anti-cyclone.

(β). The velocity in longitude is proportional to the (apparent) strength of the vortex, and for a vortex of given strength is nearly proportional to the polar distance. The rate of variation of its degrees of longitude is thus nearly uniform.

2°. Motion in latitude:—

(γ). The vortex approaches or recedes from the pole according as its density is diminishing or increasing; or, what is the same thing, according as its area (cross section) is increasing or diminishing.

(δ). The velocity in latitude is in general greater the smaller the distance from the pole.

(ϵ). A cyclone is diminishing in strength or increasing according as it is approaching the pole or receding from it. Exactly the opposite holds of an anti-cyclone.

(ζ). If a vortex approach the pole its density will fall more rapidly than that of the surrounding air, and if it be receding from the pole its density will rise faster than that of the surrounding air.

3°, combining 1° and 2°. The direction of motion of a vortex is more nearly due north and south, the closer the vortex is to the pole.

Of course, in general, we do not find a solitary vortex, but a series of them scattered over the hemisphere at once, and their mutual action must be considered in determining their motion. Thus, if there were n vortices in a circular cylinder, we must have n images at the n inverse points outside the cylinder, and at the centre a composite image, the rate of variation of whose cross section, with its sign changed, equals the algebraic sum of the rates of variation of the cross sections of the n real vortices. The com-

ponents of the velocity at any point due to such a system can be at once obtained from the principles laid down.

Before leaving the subject, I would remark that our theory asserts that a cyclone could travel from east to west only if a strong anti-cyclone were to the north of it, or a second cyclone to the south of it.

On the expression of a symmetric function in terms of the elementary symmetric functions.

By R. E. ALLARDICE, M.A.

The theorem that any rational symmetric function of n variables $x_1, x_2, \ldots x_n$ is expressible as a rational function of the n elementary symmetric functions, Σx_1, Σx_1x_2, $\Sigma x_1x_2x_3$, etc., is usually proved by means of the properties of the roots of an equation. It is obvious, however, that the theorem has no necessary connection with the properties of equations; and the object of this paper is to give an elementary proof of the theorem, based solely on the definition of a symmetric function.

It is obvious that only integral symmetric functions need be considered.

Let ${}_np_1, {}_np_2, {}_np_3, \ldots$ stand for $\Sigma x_1, \Sigma x_1x_2, \Sigma x_1x_2x_3 \ldots$, when there are n variables. If x_n vanishes, ${}_np_1, {}_np_2, {}_np_3 \ldots$ evidently become ${}_{n-1}p_1, {}_{n-1}p_2, {}_{n-1}p_3, \ldots$

Now assume that all integral symmetric functions involving not more than $(n-1)$ variables can be expressed rationally in terms of the elementary symmetric functions.

Let $f(x_1, x_2, \ldots x_n)$ be any integral symmetric function of n variables. Then $f(x_1, x_2, \ldots x_{n-1}, 0)$ is a symmetric function of $(n-1)$ variables, and, by supposition, may be expressed in terms of ${}_{n-1}p_1, {}_{n-1}p_2 \ldots$. Let its expression be $\phi({}_{n-1}p_1, {}_{n-1}p_2, \ldots {}_{n-1}p_{n-1})$.

Assume now

$$f(x_1, x_2 \ldots x_n) = \phi({}_np_1, {}_np_2 \ldots {}_np_{n-1}) + \psi(x_1, x_2, \ldots x_n),$$

where ψ is obviously a symmetric function.

Put $x_n = 0$, on both sides of this identity; then

$$f(x_1, x_2, \ldots x_{n-1}, 0) = \phi({}_{n-1}p_1, {}_{n-1}p_2, \ldots {}_{n-1}p_{n-1}) + \psi(x_1, x_2, \ldots x_{n-1}, 0);$$

and hence $\psi(x_1, x_2, \ldots x_{n-1}, 0) = 0,$

and therefore x_n is a factor in $\psi(x_1, x_2, \ldots x_{n-1}, x_n)$. Since ψ is a symmetric function, $x_1, x_2, \ldots x_{n-1}$, must also be factors; and therefore $x_1 x_2 \ldots x_n$, which is equal to ${}_n p_n$, is a factor. If this factor be divided out, the quotient will be a symmetric function, the degree of which will be less by n that of the given function. The above process may then be repeated with this quotient; and so on, till the degree is reduced to zero.

Since every (symmetric) function of a single x_1 is a function of ${}_1 p_1 (= x_1)$, it follows by induction that every symmetric function of n variables is expressible in terms of the n elementary symmetric functions.

The ordinary propositions about the weight and order of symmetric functions may easily be obtained from the above.

On laboratory work in electricity in large classes.

By Messrs A. Y. Fraser, J. T. Morrison, and W. Wallace.

Seventh Meeting, May 10th, 1889.

George A. Gibson, Esq., M.A., President, in the Chair.

Solutions of two geometrical problems.

By J. S. Mackay, LL.D.

The two problems are:—

1. *To divide a given straight line internally and externally so that the ratio between its segments may be equal to a given ratio.*

2. *To divide a given straight line internally and externally so that the rectangle under its segments may be equal to a given rectangle.*

The solutions of these problems commonly given in text-books of geometry, whether home or foreign, do not bring out so clearly as might be wished the correspondence which exists between them. Nor in the case of the second problem do the usual construction and proof for the external section run, as they ought to do, on all fours with the construction and proof for the internal section. Where correspondences between problems or between theorems exist, it cannot surely be instructive to ignore them.

It should be added that the solutions themselves are not new. The construction for Problem 1 will be found in Pappus's *Mathematical Collection*, Book III., Prop. 9; that for Problem 2 in Willebrord Snel's *Apollonius Batavus* (1608) or Edmund Halley's *Apollonii Pergaei Conica* (1710), Book VIII., Prop. 18, Scholion. Leslie, in the third and later editions of his *Elements of Geometry*, gives both Snel and Pappus's solutions of Problem 2.

Problem 1.

To divide a given straight line internally and externally so that the ratio between its segments may be equal to a given ratio.

Figure [8].

Let AB be the given straight line, K : L the given ratio.

Draw AE and BF perpendicular to AB and equal respectively to K and L. If AB is to be divided internally, AE and BF are drawn on opposite sides of it; if externally, on the same side of it. Let the straight line drawn through E and F cut AB at C.

C is the required point of division.

For triangles	ACE, BCF are similar;
therefore	AC : AE = BC : BF ;
therefore	AC : BC = AE : BF ,
	= K : L .

It may be noted that:

The straight line EF whose direction gives the point of internal or external section for the ratio K : L is the diameter of the circle whose intersection with AB gives the point of external or internal section for the rectangle K·L.

If AB, measured from A to B, is to be divided in the ratio K : L, the construction gives only one point of section, and at the same time that the ratio K : L is not equivalent to the ratio L : K.

PROBLEM 2.

To divide a given straight line internally and externally so that the rectangle under its segments may be equal to a given rectangle.

Figure [9].

Let AB be the given straight line, K·L the given rectangle.

Draw AE and BF perpendicular to AB and equal respectively to K and L. If AB is to be divided internally, AE and BF are drawn on the same side of it; if externally, on opposite sides of it. Let the circle whose diameter is EF cut AB at C.

C is the required point of division.

For triangles	ACE, BFC are similar;
therefore	$AC : AE = BF : BC$;
therefore	$AC \cdot BC = AE \cdot BF$,
	$= K \cdot L$.

It may be noted that:

The straight line EF which is the diameter of the circle whose intersection with AB gives the point of internal or external section for the rectangle K·L is the direction which gives the point of external or internal section for the ratio K : L.

If AB, measured from A to B, is to be divided in the rectangle K·L, the construction gives two points of section, and at the same time that the rectangle K·L is equivalent to the rectangle L·K.

While the preceding solutions of Problems 1 and 2 would seem to be worthy of occupying the position of "classical" solutions, it may not be out of place to call attention to two other modes of resolving Problem 2.

Figure [10].

Let AB be the given straight line, K·L the given rectangle.

Draw AE and BF perpendicular to AB and equal respectively to K and L. If AB is to be divided internally, AE and BF are drawn on the same side of it; if externally, on opposite sides of it. Let the circle whose diameter is AB cut EF at G, and let GC drawn perpendicular to EF meet AB at C.

C is the required point of division.

Join AG, BG.

Because $\angle EAG = \angle CBG$,

and $\angle AGE = \angle BGC$;

therefore triangles AEG and BCG are similar.

Hence also triangles ACG and BFG are similar;

therefore quadrilaterals ACGE and BFGC are similar;

therefore $AC : AE = BF : BC$;

therefore $AC \cdot BC = AE \cdot BF$,

$= K \cdot L$.

This method of solution, at least in the case of external section, will be found in Pappus's *Mathematical Collection*, Book VII., Prop. 157.

It may be noted that:

When K and L are equal, or when the rectangle K·L is transformed into a square, in the one case EF is parallel to AB, and in the other case EF passes through the middle point of AB; or, as was previously remarked, EF passes through the point where AB is divided externally or internally in a ratio of equality. The constructions for these two cases usually given in foreign text-books (see Rouché et De Comberousse's *Traité de Géométrie*, Livre III., §§ 255, 257, or Vacquant's *Cours de Géométrie Elémentaire*, Livre III., §§ 310, 311) are now seen to be perfectly correspondent, and to be merely modifications of that given long ago by Pappus.

The following method of solution was communicated to me by Mr A. Y. Fraser.

Figure [11].

Let AB be the given straight line, DE·EF the given rectangle.

Describe any circle passing through D and F, and let O be its centre. In this circle place GH equal to AB. With O as centre describe a circle to touch GH, and through E draw the chord KL touching the second circle.

Then $KL = GH = AB$,

and $KE \cdot EL = DE \cdot EF$.

Hence the required segments of AB are equal to KE and EL.

For comparison with the above the constructions given in Playfair's *Elements of Geometry*, VI., 28, 29 are annexed.

Figure [12].

Let AB be the given straight line, C a side of the square to which

the rectangle under the segments (internal or external) of AB is to be equal.

Case 1. Bisect AB in D; draw DE perpendicular to AB and equal to C. Produce ED to F so that EF be equal to AD or DB; with centre E and radius EF describe a circle cutting AB in G.

G is the required point of division.

Case 2. Bisect AB in D; draw BE perpendicular to AB and equal to C. Join DE; with centre D and radius DE describe a circle cutting AB produced in G.

G is the required point of division.

Kötter's synthetic geometry of algebraic curves.

By NORMAN FRASER, B.D.

In the following paper I propose to give a short account of Dr Ernst Kötter's purely geometrical theory of the algebraic plane curves. This theory is developed in a treatise* which, in 1886, gained the prize of the Berlin Royal Academy; but the contents of my paper are also partly drawn from a course of lectures delivered by Dr Kötter in the University of Berlin, W.S. 1887-88.

The method followed is that of involutions, and we might pass at once to a discussion of these; but it may be of interest to call attention, first of all, to one of the most remarkable features of the theory—Dr Kötter's treatment of imaginary points and lines. He undertakes to show that the projective relationship between two aggregates may be extended so as to embrace their imaginary elements, and, hence, that imaginary elements may take their place with real elements in the theory of involutions. One method of establishing this result has been given by von Staudt;† but it travels beyond the limits of Plane Geometry. Many writers on the subject simply assume the point at issue; a course which has its advantages, and may be recommended to any reader who should chance to find

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* *Grundzüge einer rein geometrischen Theorie der algebraischen ebenen Curven.* (*Transactions of the Royal Academy of Science.* Berlin, 1887. Also published separately in the same year.)

† *Beiträge zur Geometrie der Lage.* Nürnberg, 1856-60.

the following condensed and necessarily imperfect account of Dr Kötter's method unintelligible.

IMAGINARY ELEMENTS.

An elliptic involution has, we know, no real double elements. Imaginary points in a line are defined as the double points of the various elliptic involutions which may be formed of the real points of the line. Each involution thus yields a pair, an imaginary point and its conjugate, which, like the involution itself, are completely determined when two real pairs of the involution, say AA′ and BB′, are given. These pairs will divide one another, B and B′ lying the one in the finite, and the other in the infinite line AA′. We denote the two imaginary points which they determine by writing them down in order, as ABA′B′. It remains to distinguish between the two imaginary points thus denoted, and, in doing so, we follow von Staudt. To prepare the way for this distinction let us glance at the case of a hyperbolic involution. Here, also, we may determine the two real double points by two pairs of the involution, AA′ and BB′. Now, of these double points, one lies to the *right*, and the other to the *left* of the centre of the involution, and, to give effect to this distinction, we may indicate the former by naming our four points *from left to right*, say ABB′A′, and the latter by naming them from *right to left*, AA′B′B, or A′B′BA. This is what we agree to do in the case of the elliptic involution; we assign to each imaginary double point a particular direction, which is indicated by the order in which we name the four points which determine the involution; and if we denote the one by ABA′B′, we denote the other by AB′A′B, or, which is the same thing, by B′A′BA. Similarly, if CC′, DD′ be any other pairs of the same involution, out imaginary points may equally well be denoted by CDC′D′ and CD′C′D.

In exactly the same way imaginary lines, through a real point, are defined as the double rays of an elliptic involution in a pencil, and are denoted by *aba′b′*, *ab′a′b*, respectively.

An imaginary line is said to contain an imaginary point, if it be possible to represent the one by *aba′b′* and the other by ABA′B′, where *a*, *b*, *a′* and *b′* are the lines joining A, B, A′ and B′ to O, the real point of the imaginary line. (In this case *ab′a′b* will evidently contain AB′A′B.)

To find the imaginary line joining two imaginary points which lie in two real lines, we find two representations of them, ABA′B′ and CDC′D′, which are in perspective; that is, such that CA, DB, C′A′, and D′B′ meet in one point. This point will be the real point of the imaginary line required. It can be proved that only one such point can be found, so that the problem is a completely determinate one. Similarly, two imaginary lines intersect in a perfectly definite imaginary point.

From any imaginary point outside a real line we may project its imaginary points—of which there is a twofold infinity—into the real points of the plane. We are thus able to represent the points, real and imaginary, of a line by the real points of the plane: the imaginary points being represented by points outside of the line, while the real points of the line represent themselves. Again, the points of another line may be represented by a different arrangement of the points of the plane, or, which is the same thing, by the points of a superimposed plane; while the real and imaginary rays of a pencil may be represented by the real lines of a plane. We are now able to investigate the nature of the projective relationship between imaginary elements in two aggregates by observing the real points and lines which represent these imaginary elements. In the case of imaginary, as in that of real elements, Dr Kötter defines the projective relationship as the result of a series of perspective relationships: that is, two aggregates are projective when they can be connected by a series of aggregates, of which any consecutive pair are in perspective. Accordingly our task reduces itself to that of observing the relation subsisting between the points and lines of two planes, when the aggregates of imaginary elements, which they represent, are in perspective.

This relation is somewhat complicated, and is only loosely described when we say, that when a point or line in the one plane describes or envelopes a conic, the corresponding point or line in the other plane describes or envelopes a corresponding conic; but it establishes conclusively that in the case of imaginary, as in that of real elements, the projective relationship is *continuous*, *unambiguous*, and *determined by three correspondences.*

We are thus able to carry over to imaginary elements all the known projective properties of real elements. Hereafter, when we

speak of ranges or pencils, these are to be understood as containing both real and imaginary elements.

Involutions of Various Degrees.

Involutions of the Second Degree.

If, in two projective aggregates on the same base, every element has the same correspondent, whether considered as belonging to the first or the second aggregate, then each element makes, with its correspondent, a pair of what is termed an involution of the second degree. For example, if

$$A_1B_1C_1A_2B_2C_2 \ \dots \barwedge A_2B_2C_2A_1B_1C_1 \ \dots$$

then A_1A_2, B_1B_2, C_1C_2 are pairs of an involution. It is easy to prove that, if the required property holds for any one element, say C_1, it will hold for all; that is, that if we can prove

$$A_1B_1C_1C_2 \ \dots \barwedge A_2B_2C_2C_1 \ \dots$$

then A_1A_2, B_1B_2, C_1C_2 will be an involution as above.

A second degree involution may be generated in the following way:—Let A_1A_2, B_1B_2, C_1C_2 be an involution. Then

$$\begin{aligned} A_1B_1C_1C_2 &\barwedge A_2B_2C_2C_1 \ \dots \\ &\barwedge B_2A_2C_1C_2 \ \dots \end{aligned}$$

that is, C_1 and C_2 are double elements of a duplex aggregate

$$A_1B_1X \ \dots \barwedge B_2A_2Y \ \dots$$

Again, if E_1E_2 are another pair of the involution,

$$\begin{aligned} A_1B_1E_1E_2 &\barwedge A_2B_2E_2E_1 \ \dots \\ &\barwedge B_2A_2E_1E_2 \ \dots \end{aligned}$$

that is E_1 and E_2 are double elements of another duplex aggregate

$$A_1B_1X' \ \dots \barwedge B_2A_2Y' \ \dots$$

Similarly any other pair of our involution can be generated as double elements of a duplex aggregate of the form

$$A_1B_1 \ \dots \barwedge B_2A_2 \ \dots$$

So we may generate our involution by taking

$$A_1B_1M \ \dots \barwedge B_2A_2N \ \dots$$

where M is fixed while N varies, and by taking the double elements of the various duplex aggregates so formed. (To give the pair

A_1A_2, N must be A_2; and to give B_1B_2, N must be B_2.) Hence corresponding to successive pairs

$$A_1A_2 \quad B_1B_2 \quad C_1C_2 \quad D_1D_2 \quad E_1E_2 \quad \dots$$

we have a series of values of N :— $\quad A_2 \quad B_2 \quad N_1 \quad N_2 \quad N_3 \quad \dots$

Similarly, if instead of M we take M′, we get for N :— $\quad A_2 \quad B_2 \quad N_1' \quad N_2' \quad N_3' \quad \dots$

and if we take M″ we get :— $\quad A_2 \quad B_2 \quad N_1'' \quad N_2'' \quad N_3'' \quad \dots$

All these aggregates are called "characteristic" ranges or pencils of the involution. It can be proved that they are all projective to one another for a given involution, and we are thus entitled to call the involution itself projective to any of them, and write

$$\begin{array}{cccccc} A_1A_2, & B_1B_2, & C_1C_2, & D_1D_2, & E_1E_2 & \dots \\ \barwedge\ A_2 & B_2 & N_1 & N_2 & N_3 & \dots \end{array}$$

Further, if two involutions have projective characteristic ranges, they are said to be projective to one another.

Involutions of the Third Degree.

Suppose we take an involution of the second degree,

$$A_1A_2, \ B_1B_2, \ M_1M_2 \ \dots$$

and set it projective to a range on the same base $\quad B_3 \quad A \quad N \quad \dots$ (where N is variable); then points which are common to the involution and range for a given position of N, make up a group of what is termed an involution of the third degree. If, when N is N_1, our involution takes the form $\quad A_1A_2, \ B_1B_2, \ M_1M_2, \ P_1P_2, \ Q_1Q_2, \ R_1R_2 \ \dots$
and our range takes the form $\quad B_3 \quad A_3 \quad N_1 \quad P_2 \quad Q_2 \quad R_2 \quad \dots$
then P_2 Q_2 and R_2 are members of the particular group of the third degree involution corresponding to N_1. It can be proved that a second degree involution and a projective range have always three points common; thus our third degree involution is made up of groups of *three* elements. As before, the range described by N, while M_1M_2 are fixed, is called the characteristic range (or pencil, when we are dealing with pencils), and we have

$$\begin{array}{lllll} A_1A_2A_3, & B_1B_2B_3, & C_1C_2C_3, & D_1D_2D_3 & \dots \\ \barwedge\ A_3 & B_3 & N_1 & N_2 & \dots \\ \barwedge\ A_3 & B_3 & N_1' & N_2' & \dots \ \text{\&c.} \end{array}$$

The A's and B's have no properties not shared by other groups of the involution: if $K_1K_2K_3$, $L_1L_2L_3$ are any two groups, the involution may be generated from $\quad K_1K_2, \ L_1L_2 \ \dots \barwedge L_3 \ K_3 \ \dots$

If two groups contain the same element it will be common to all groups.

If an involution should contain two threefold elements (that is, two groups of the form DDD, D'D'D'), all other groups will be regular. If it should contain one threefold element, there will be two double elements (two groups of the form PPP_1, QQQ_1); if no threefold element, there will be four double elements.

Involutions of the n^{th} Degree.

We pass now to involutions of the n^{th} degree. If we take two projective involutions of the $\overline{n-m}$ and m^{th} degree respectively, viz. :—

$$\begin{array}{llllll} & A_1A_2A_3 & \dots\ A_{n-m}\ , & B_1B_2B_3 & \dots\ B_{n-m}\ , & \dots \\ \barwedge & B_{n-m+1} & \dots\ B_n\ , & A_{n-m+1} & \dots\ A_n\ , & \dots \end{array}$$

and make a variable group of the latter correspond to a fixed group of the former, we shall obtain a series of groups of n elements common to the two involutions, which will together make up what is termed an involution of the n^{th} degree. This involution will be projective to the m^{th} degree involution traced out by the variable group.

The involution is determined by any two groups. A given element of the base belongs to one group of the involution or else to all; in which last case the involution may be broken up into that element and an involution of the $\overline{n-1}^{st}$ degree.

If an involution of the n^{th} degree contains two n^{fold} elements, all other groups are regular; if one n^{fold} element, at most $n-1$ groups will contain double elements. (We make use of this in the theory of polars.)

Rank in Involutions.

Before introducing a new distinction, that of "rank," we must define "involution-pencils" and "nets." Suppose we have an involution of the n^{th} degree V_1, V_2, V_3 ... and a fixed group U which does not belong to it, then the involution U, V will, while V traces out the involution V_1, V_2, V_3 ..., trace out an "involution-pencil" with centre U. We denote this pencil by $U(V_1V_2V_3 \dots)$, and we say that it is in perspective with the involution V_1,V_2,V_3 .... All involution-pencils which are in perspective with the same or projective involutions are said to be projective to one another.

Any three groups U V W which do not belong to the same involution determine an "involution-net" of the second dimension, which is made up of all the groups of all the involutions of the pencil U(V,W ...). This net contains every involution which is determined by any two of its groups. The net of the second dimension is strictly analogous to a plane, an involution corresponding to a line, and a group to a point. Any four groups U V W X which do not belong to the same net of the second dimension, determine a net of the third dimension, which is made up of all the groups of all the involutions determined by U and each of the groups of the second dimension net VWX. It corresponds to space of three dimensions. Generally, any $\mu+1$ groups (of the same degree, of course) which do not belong to the same net of the $\overline{p-1}^{st}$ dimension, determine a net of the μ^{th} dimension. It may be defined as follows: Let U,V,W be an involution, U being a fixed group; let V describe a net of the $\overline{u-1}^{st}$ dimension, then W will describe a net of the μ^{th} dimension. This net contains entirely all nets of lower dimensions determined by any of its groups. All contained nets of the $\overline{\mu-1}^{st}$ dimensions which have in common a given net of the $\overline{\mu-2}^{nd}$ dimension make up what is termed a net-pencil.

Involutions of the Second Rank.

Now let us take in the same net of the second dimension two projective, but not perspective involution-pencils with centres at U and V, and let corresponding involutions of the pencils have in common the groups W_1, W_2, W_3 ... ; then $U(W_1W_2W_3 \ldots) \barwedge V(W_1W_2W_3 \ldots)$ generate what is termed an involution of the second rank. Evidently such an involution (to be distinguished from all involutions hitherto discussed, which were of first rank) corresponds exactly to a conic in plane geometry; it will contain U and V, and be determined by any five groups. Moreover, it has two groups in common with any first rank involution in the same second dimension net.

Involutions of the Third Rank.

Again, if, in a net of the third dimension, we take three projective pencils whose axes are the involutions U_1, U_2; U_2, U_3; viz.,

$$U_1U_2(U_4U_5U_6 \ldots U)$$
$$\barwedge U_2U_3(U_4U_5U_6 \ldots U)$$
$$\barwedge U_3U_1(U_4U_5U_6 \ldots U);$$

then pencils will intersect in a series of groups $U_4\ U_5\ U_6\ \dots$ which make up what is termed an involution of the third rank. As we see, it is determined by six groups. An involution of the third rank has three groups in common with any second dimension net contained in the same third dimension net.

Involutions of the μ^{th} Rank.

Generally, an involution of the μ^{th} rank is determined by $\mu + 3$ groups, which all lie in the same net of the μ^{th} dimension, and is generated by the intersection of corresponding nets of μ projective $\overline{\mu - 1}^{st}$ dimension net-pencils :—

$$\begin{array}{ll} & U_2U_3 \dots U_\mu \quad (U_{\mu+1}, U_{\mu+2}, U_{\mu+3} \dots U) \\ \barwedge & U_3U_4 \dots U_\mu U_1 (U_{\mu+1}, U_{\mu+2}, U_{\mu+3} \dots U) \\ \barwedge & U_4U_5 \dots U_1\, U_2 (U_{\mu+1}, U_{\mu+2}, U_{\mu+3} \dots U) \\ & \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \barwedge & U_1U_2 \dots U_{\mu-1} (U_{\mu+1}, U_{\mu+2}, U_{\mu+3} \dots U) \end{array}$$

The rank of an involution cannot be higher than its degree. We shall call an involution of the m^{th} degree and μ^{th} rank an (m, μ) involution. A simple range or pencil may be looked upon as a $(1, 1)$ involution.

An (m, μ) involution has with a projective (n, ν) involution on the same base $m\nu + n\mu$ elements in common.

Theory of Curves.

Conics.

We begin with the conic. A conic is generated by the intersection of corresponding rays of two projective pencils with centres at real or imaginary points; that is, if we have two pencils $P(R\ S\ T \dots)$ $\barwedge$ $Q(R\ S\ T \dots)$ where R S and T are not in the same straight line, then corresponding rays will meet in a conic. A conic is thus seen to be determined by five points, and the same conic is generated whatever points upon it be taken as the centres of the generating pencils. The tangent at P is the ray in the pencil P which corresponds to QP in Q; similiarly the tangent at Q is the ray in Q which corresponds to PQ in P.

Conic-Pencils.

Two conics meet in four points, through which an infinite number of conics may be drawn, constituting what is termed a conic-pencil. This, however, is otherwise defined. Let two conics, K_1^2

and K_2^2, have a common point at P.* Let Q be a point on K_1^2 and R a point on K_2^2. Then K_1^2 may be generated by pencils at P and Q :— $p_1p_2p_3 \ldots \barwedge q_1q_2q_3 \ldots$
and K_2^2 by pencils at P and R :— $p_1p_2p_3 \ldots \barwedge r_1r_2r_3 \ldots$
Then $q_1q_2q_3 \ldots \barwedge r_1r_2r_3 \ldots$ will generate another conic K^2, which will pass through Q and R; also through all the intersections of K_1^2 and K_2^2 except P, since at each of these intersections corresponding rays p_n and q_n, p_n and r_n and therefore q_n and r_n meet. Now take on K^2 a series of points S T, &c.; then K^2 may be generated by any two of the pencils

$$\begin{array}{ll} q_1q_2q_3 \ldots & \qquad\text{(where } q_1r_1s_1t_1 \ldots \\ \barwedge\ r_1r_2r_3 \ldots & \qquad q_2r_2s_2t_2 \ldots \\ \barwedge\ s_1s_2s_3 \ldots & \qquad q_3 \ldots \ldots \\ \barwedge\ t_1t_2t_3 \ldots & \qquad\text{are evidently also projective pencils).} \\ \ldots\ \ldots\ \ldots & \end{array}$$

Now the pencil $p_1p_2p_3 \ldots$ will generate along with any one of these pencils a conic which contains P and all the other intersections of K_1^2 and K_2^2; and all such conics together constitute a "conic-pencil." We call the points common to the conics the "ground-points" of the pencil. We say that the conic-pencil is projective to the ray-pencil made up of the tangents at any of the ground-points to the various conics of the pencil. It is also projective to any of the ray-pencils $q_1r_1s_1t_1 \ldots$
$q_2r_2s_2t_2 \ldots$ &c.

Any straight line meets a conic-pencil in an involution of the second degree. For, take any straight line l; it will meet the pencil in points formed by taking points common to the range

$$\underline{l(p_1p_2p_3 \ldots)}$$

and the various projective ranges

$$\begin{array}{l} l(q_1q_2q_3 \ldots) \\ l(r_1r_2r_3 \ldots) \\ l(s_1s_2s_3 \ldots) \\ \ldots\ \ldots\ \ldots \end{array}$$

The p's and q's will give a pair Q_1Q_2; the p's and r's R_1R_2, and so on; and from the theory of involutions it follows that Q_1Q_2 R_1R S_1S_2 T_1T_2 are pairs of an involution of the second degree, of which each of the ranges $l(q_1r_1s_1 \ldots)$, &c., is a characteristic range. l thus meets the conic-pencil in a projective involution.

* The diagram, which is very simple, may be supplied by the reader.

Similarly any conic will meet the conic-pencil in a projective involution of the fourth degree.

Through a given point (not one of the ground-points) one and only one conic of the pencil passes. For let P be the point: any line l through P will meet the pencil in a second degree involution, in which there will be one point P^1 conjugate to P. Then as l revolves round P, P^1 will describe the conic of the pencil which passes through P.

We may define "conic-nets" just as in the case of involutions. Any three conics which do not belong to the same pencil determine a net of the second dimension. For example, all conics which pass through three given points make up a conic-net of the second dimension. Similiarly, all which pass through two points make up a net of the third dimension, and so on; though it does not follow that all the conics of nets of the second or third dimension pass through three or two points respectively.

Generation of Conics by Involutions.

So far we have looked on conics as generated by two pencils, or (for we may so express it) by a pencil and a (1, 1) ray-involution. They may also be generated by the aid of involutions of higher degree and rank. If we cut a conic by the rays of a pencil from a point P which does not lie upon the conic, the intersections of each ray will be projected from a point Q on the conic into a pair of a ray-involution of the second degree with centre Q. If Q be not on the conic, the involution will be of the second degree and second rank. In either case the involution at Q will be projective to the pencil at P. So we can generate a conic by means of a simple pencil and a projective (2, 1) or (2, 2) involution. From either of these modes of generation we can ascertain the number of points of intersection of two conics. Take the former. Let K_1^2 be generated in the ordinary way by two pencils at R and S; K_2^2 may be generated by the pencil R and a (2, 1) involution at one of its points Q; also by the pencil S and another (2, 1) involution at Q projective to the first. Evidently any ray QX common to these two involutions will meet the corresponding rays of R and S at a point X which is common to both conics. But two projective (2, 1) involutions have four common elements; hence two conics intersect in four points, real or imaginary.

Cubics.

Suppose we have a fixed point P and through it a pencil $p_1p_2p_3$ ... ; further, four points $P_1P_2P_3P_4$ and through them a projective conic-pencil L_1^2, L_2^2, L_3^2 ... then the intersections of corresponding members of $p_1\ p_2\ p_3\ \ldots \barwedge L_1^2, L_2^2, L_3^2 \ldots$ generate a "cubic" or curve of the third degree. Any line l will meet this curve in points common to the range and involution

$l(p_1p_2p_3\ \ldots\ ) \barwedge l(L_1^2, L_2^2, L_3^2\ \ldots\ )$ that is in a group of three points.

P is evidently a point on the curve. Let L^2 be the conic of the pencil which passes through P, then the corresponding ray p will be the tangent at P to the cubic.

Cubic-Pencils.

Suppose we take two cubics K_1^3 and K_2^3. Let P be any common point, and let $p_1p_2p_3$ ... be a pencil at P.

Then K_1^3 may be generated by $p_1p_2p_3\ \ldots \barwedge L_1^2,\ L_2^2,\ L_3^2\ \ldots$
and K_2^3 may be generated by $p_1p_2p_3\ \ldots \barwedge M_1^2, M_2^2, M_3^2\ \ldots$
These two conic-pencils determine an "array" (Schaar) of conics made up of projective pencils

$$\begin{array}{llll} L_1^2, & L_2^2, & L_3^2 & \ldots \\ M_1^2, & M_2^2, & M_3^2 & \ldots \\ N_1^2, & N_2^2, & N_3^2 & \ldots \\ Q_1^2, & Q_2^2, & Q_3^2 & \ldots \\ \ldots & \ldots & \ldots & \end{array}$$

where $L_1^2, M_1^2, N_1^2, Q_1^2\ \ldots$ belong to a pencil which is projective to $L_2^2, M_2^2, N_2^2, Q_2^2$ and so on.

Then the cubic generated by $p_1p_2p_3$ ... and the various pencils of our array constitute what is termed a "cubic-pencil." Any line will meet this pencil in an involution of the third degree. Any point in the plane determines one and only one cubic of the pencil.

Cubics may also be generated by a pencil and a (3, 3) ray-involution.

Curves of the Fourth and Higher Degrees.

A curve of the fourth degree may be generated by a pencil and a projective cubic-pencil; also by two projective conic-pencils.

And generally, a curve of the n^{th} degree may be generated by two projective curve-pencils of the $\overline{n - m}^{\,th}$ and m^{th} degree respectively.

That is, K^n is generated by $\quad K_1^{n-m},\ K_2^{n-m},\ K_3^{n-m}\ \ldots$
$\qquad\qquad\qquad\qquad\qquad\quad \barwedge\ K_1^m\ ,\ K_2^m\ ,\ K_3^m\ \ldots$

It will meet any line in the plane in general in n points.

Generation of Curves by Involutions.

A curve of the n^{th} degree may also be generated by the aid of involutions. Suppose we have a pencil P and an (m, μ) ray-involution at a point Q. Then, if we set

$$p_1 p_2 p_3 \ldots \barwedge q_m^1 q_m^2 q_m^3 \ldots$$

the intersections of each ray p_r with the corresponding group of rays q_m^r will develope a curve of the m^{th} degree. It can be shown, however, that the locus developed in this way includes besides our required curve the μ^{fold} line PQ (compare the familiar fact that two perspective pencils develope not only the base of perspective, but also the line joining the two centres). This has to be subtracted in order to give us our curve K^m. Thus, to find the number of intersections of any line with our curve. Let l be any line in the plane. It may be generated by $p_1 p_2 p_3 \ldots$ and a perspective pencil $q_1^1 q_1^2 q_1^3 \ldots$ at Q. Then all intersections of l with K^m will be given by rays common to

$$q_1^1 q_1^2 q_1^3 \ldots \barwedge q_m^1 q_m^2 q_m^3 \ldots$$

But these are $(1, 1)$ and (m, μ) involutions; hence they have $m+\mu$ common rays. Of these, however, μ coincide in PQ, and we are left with the result that l meets K^m in m points.

It can be shown that Q is an $\overline{m-\mu}^{fold}$ point on K^m: if $\mu = m$, Q does not lie on the curve.

Points common to two Curves.

To find the number of points common to K^m and K^n. Let K^m be generated by $\quad p_1 p_2 p_3 \ldots \barwedge q_m^1 q_m^2 q_m^3 \ldots \quad (m, \mu)$
and K^n by $\quad p_1 p_2 p_3 \ldots \barwedge q_n^1 q_n^2 q_n^3 \ldots \quad (n, \nu)$
then points common to K^m and K^n are projected into rays common to $q_m^1 q_m^2 q_m^3 \ldots \barwedge q_n^1 q_n^2 q_n^3$. Of these there are $m\nu + n\mu$; but $\mu\nu$ of them are accounted for by the coincidence of the μ^{fold} ray QP with the ν^{fold} ray QP. Hence K^m and K^n have in common, outside of Q, $m\nu + n\mu - \mu\nu$ points. In the general case when Q does not lie on either curve, $m=\mu$ and $n=\nu$; thus in general two curves of the m^{th} and n^{th} degree have mn points in common. They cannot have more unless K^m and K^n have in common some curve of a lower degree, and take the form $K^r\ K^{m-r}$ and $K^r\ K^{n-r}$.

Propositions on Common Points.

If, of the p^2 intersections of two curves, K_1^p and K_2^p, pm lie on K^m, the remaining p^2-pm shall lie on some curve K^{p-m}. Take the pencil K_1^p, K_2^p and find the curve K_3^p in it which holds an additional point Q of K^m. Then K_3^p and K^m have $pm+1$ points in common (namely, pm of the ground-points of the pencil, and Q), and, therefore, must have in common some curve of lower degree. Let K_3^p be of the form K^rK^{p-r} and K^m of the form K^rK^{m-r}. (If $r=m$ our proposition is proved). Again, K^rK^{p-r} has with K_1^p p^2 common points, of which pm lie in K^rK^{m-r}. Therefore K^{p-r} has with K_1^p p^2-pr common points, of which $pm-pr$ lie on K^{m-r}. That is, K^{p-r} and K^{m-r} have p^2-pr instead of $(p-r)(m-r)$ points in common, and hence must be of the form K^sK^{p-r-s} and K^sK^{m-r-s}. And so we can proceed until $r+s+\ \dots\ =m$, proving that K_3^p is of the form K^mK^{p-m}. That is, the p^2-pm ground-points of the pencil which do not lie on K^m lie on a curve K^{p-m}. Q.E.D.

We can prove the more general proposition: if K^m and K^n have mn points in common, of which pn lie on K^p, the remaining $(m-p)n$ shall lie on some curve K^{m-p}.

All curves of the p^{th} degree which contain $3p-1$ fixed points belonging to a cubic, shall also contain another point dependent on these. For, let K_1^p and K_2^p have in common $(3p-1)$ points of K^3. K_1^p and K_2^p will each cut K^3 in one additional point: let these be P and Q. We wish to prove that P and Q are identical.

Through P take any line a, cutting K^3 in P_1 and P_2, then K_2^pa (a curve of the $\overline{p+1}^{st}$ degree) cuts K^3 in $3p+3$ points: $3p$ of these (viz., the $3p-1$ fixed points and P) lie on K_1^p; therefore, by the proposition enunciated above, the remaining three, Q P_1 and P_2, lie in a straight line. But this can only be the line a, and as a meets K^3 in three points only, P and Q must be identical. Hence every curve of the p^{th} degree, which passes through the $3p-1$ points, passes through P. Q.E.D.

When $p=3$ we get the following proposition:—Of the nine ground-points of a cubic-pencil only eight are independent, and every cubic which contains these must contain the ninth also. It follows from this that nine independent points are necessary and sufficient to determine a cubic. The general proposition, of which that proved above is a particular case, runs: all curves of the

p^{th} degree which contain $pn - \frac{1}{2}(n-1)(n-2)$ points of a curve of the n^{th} degree, shall contain in addition $\frac{1}{2}(n-1)(n-2)$ points dependent on these. When $p = n$ we get the proposition: of the n^2 ground-points of a pencil of the n^{th} degree only $n^2 - \frac{1}{2}(n-1)(n-2)$, that is, $\frac{1}{2}(n^2 + 3n - 2)$ are independent. Hence a curve of the n^{th} degree is determined by $\frac{1}{2}n(n+3)$ independent points.

Multiple Points.

A line through an m^{fold} point on a curve can meet the curve at $n - m$ additional points at most.

We shall look first at double points. Let K^n have a double point at D. Suppose K^n generated by

$$d_1 d_2 d_3, \ldots \barwedge K_1^{n-1}, K_2^{n-1}, K_3^{n-1} \ldots$$

d_1 is to meet K^n, and, therefore, K_1^{n-1} at only $n - 2$ points outside of D: $\therefore$ K_1^{n-1} must pass through D. Similiarly for K_2^{n-1}, &c. So, in order that D shall be a double point, K^n must be capable of being generated by a ray-pencil and a curve-pencil both passing through D.

Let $c_1 c_2 c_3 \ldots$ be the tangents to the K^{n-1}'s at D; then tangents to K^n at D will be the double rays of $c_1 c_2 c_3 \ldots \barwedge d_1 d_2 d_3 \ldots$ Let these be t and t^1; if they are real and distinct, D is a *node* on K^n; if real and coincident, D is a *cusp;* if imaginary, D is a *conjugate point.*

In order to be a triple point of K^n, D must be a double point of K_1^{n-1}, K_2^{n-1}, &c.; and, generally, in order to be an m^{fold} point of K^n, it must be an $\overline{m-1}^{fold}$ point of the $\overline{n-1}^{st}$ degree pencil.

Again, multiple points may, as we saw, be discussed in connection with the other mode of generating curves: in order that D should be an m^{fold} point, the intersections of any pencil $p_1 p_2 p_3 \ldots$ with K^n must determine an $(n, n-m)$ involution at D.

Polar Curves.

If from a point P a line l be drawn to meet a curve K^n, then the locus of the double elements of the involution $(P)^n$, $l(K^n)$, as l revolves round P, is called the "first polar" of P with respect to K^n. That is, we take the involution of the n^{th} degree determined by two groups, of which one is the n^{fold} element P, and the other the intersections of l with K^n; we find all the groups which take the form $X^2 X_1 X_2 \ldots X_{n-2}$, and the locus of X as l revolves is our

required polar curve. There are in general $n-1$ such double points in such an involution, and the curve generated is of the $\overline{n-1}^{st}$ degree. We shall denote it by P^{n-1}. When l touches K^n at a point Q, then $l(K^n)$ itself takes the form $Q^2Q_1Q_2 \ldots Q^n$. Hence Q lies on the polar; or, the first polar of P passes through all the points of contact of tangents from P to K^n.

The polar of P with respect to P^{n-1}, is called the second polar with respect to K^n, and is denoted by P^{n-2}; finally, P^1 is called the polar line of P. If P lies on K^n it will also lie on P^{n-1}, P^{n-2}, &c., and P^1 will be the tangent at P to K^n and its successive polars.

The polars of the points of a range with respect to a curve will constitute a curve-pencil projective to the range.

The polars of a point with respect to the curves of a pencil will constitute a projective curve-pencil.

If through P we take n lines, $a_1a_2a_3 \ldots a_n$, and let these along with K^n determine an n^{th} degree pencil, P shall have the same polars with respect to all the curves of the pencil. For, through P take a line l; it will cut the pencil $a_1a_2 \ldots a_n$, K^n, $K_1{}^n$, $K_2{}^n$ ... in an involution $(P)^n$, $l(K^n)$, $l(K_1{}^n)$, $l(K_2{}^n)$ ... Accordingly, all the involutions $(P)^n$, $l(K^n)$; $(P)^n$, $l(K_1{}^n)$; $(P)^n$, $l(K_2{}^n)$, &c., are one and the ame involution, and hence the polars, which are defined by the aids of these involutions, are also identical. Q.E.D.

If the first polar of P contains a point Q, the polar line of Q shall contain P. For, take through P n lines $a_1a_2 \ldots a_n$. In the pencil $a_1a_2 \ldots a_n$, K^n take the particular curve $K_0{}^n$ which contains Q. Then the polar line $Q_0{}^1$ will be the tangent at Q to $K_0{}^n$. Again, since all polars of P are identical, P^{n-1} is the first polar of $K_0{}^n$; and, therefore, its intersections with $K_0{}^n$ give points of contact of tangents from P. But Q lies in both P^{n-1} and $K_0{}^n$. $\therefore$ PQ is the tangent to $K_0{}^n$ at Q, that is, $Q_0{}^1$ is PQ. Again, the polar of Q, with respect to $a_1a_2 \ldots a_n$, is easily seen to be a curve made up of double rays of the ray-involution $(PQ)^n$, $a_1a_2 \ldots a^n$; that is, it consists of $n-1$ rays through P. Similarly for successive polars, until the polar line of Q, with respect to $a_1a_2 \ldots a_n$, is seen to be a line through P. Now we have proved that the polar line of Q, with respect to *two* curves of the n^{th} degree pencil, is a line through P; therefore this is true for all curves of the pencil, that is, Q^1 contains P. Q.E.D.

The extended form of this proposition is: if P^{n-m} hold Q, Q^m shall hold P.

Plücker's First Equation.

A curve and its first polar meet in general in $n(n-1)$ points; hence $n(n-1)$ tangents can in general be drawn from a given point to the curve. But in each double point of K^n two of the points of intersection of K^n and P^{n-1} coincide; and, moreover, this point is no longer a point of contact in the usual sense; hence, for every double point of K^n two of the possible tangents from P disappear. Similarly, for every cusp three points of contact disappear. Hence we get Plücker's first equation for a curve which does not contain singularities of a higher order:—$k = n(n-1) - 2d - 3s$ (k being the "class" of the curve, d the number of its nodes, and s of its cusps).

Dual Treatment of Curves as Envelopes.

Again, we might prove that a curve K^n could equally be developed as an envelope by means of ranges and point-involutions, and we could then dualize all the results obtained. For example, in Plücker's equation n becomes k, d becomes t (the number of double tangents), and s becomes w (the number of points of inflection), and we get $n = k(k-1) - 2t - 3w$.

The above may serve as illustrations of the application of Dr Kötter's methods to the theory of curves.

Eighth Meeting, June 14th, 1889.

GEORGE A. GIBSON, Esq., M.A., President, in the Chair.

Sur une propriété projective des sections coniques.

Par M. PAUL AUBERT.

Théorème.—*On considère tous les cercles σ passant par deux points fixes, dont l'un c est sur la circonférence d'un cercle donné s, et l'autre d sur une droite donnée l. Chacun des cercles σ rencontre la droite l en un second point d', et la circonférence s en un second point c': La droite c'd' passe par un point fixe i de la circonférence s, quel que soit le cercle σ considéré.*

Nous pouvons en effet définir tous les cercles σ au moyen d'un troisième point qui décrirait la circonférence s. Or, quelle que soit la position c' de ce point, la droite indéfinie $c'd'$ forme avec cc' un angle qui, compté à partir de cc' dans un sens déterminé, celui des aiguilles d'une montre par exemple a toujours la même valeur : c'est l'angle cdd' si le point d est extérieur au cercle s, c'est son supplément si le point d est intérieur à ce cercle. D'ailleurs le sommet c' de cet angle est sur la circonférence s, et son côté $c'c$ passe par un point fixe de cette circonférence. Donc l'autre côté $c'd'$ rencontre bien la circonférence en un point fixe i.

Si l'on mène par le point où la droite cd rencontre la circonférence s une parallèle a la droite l, elle passera manifestement par le point i, car on peut considérer la droite indéfinie cd comme la circonférence de rayon infini à laquelle correspond un point d' infiniment eloigné sur la droite l; gi est donc une position particulière de la droite mobile $c'd'$.

Cela posé, soumettons la figure à une transformation projective. Le cercle s deviendra une conique S; les cercles σ donneront le faisceau des coniques Σ passant par quatre points fixes A, B, C, D dont les trois premiers sont sur la conique S, la droite AB provenant de la droite de l'infini de la première figure. La droite l donnera une droite L passant par le point D. La propriété démontrée précédemment étant projective subsistera : *la droite* C'D' *qui joint le second point d' intersection de chaque conique* Σ *avec la droite* L *au quatrième point commun aux deux coniques* S *et* Σ *coupera la conique* S *en un point fixe* I.

Voyons ce que devient la construction qui donnait le point i. A la droite gi parallèle à l correspondra une droite rencontrant L sur la droite AB. Il suffira donc de joindre le point d' intersection de L avec AB au point G où la droite CD coupe la conique S, pour obtenir le point I de cette conique.

Actuellement, rien ne distingue le point C des points A et B. On pourrait par une nouvelle transformation homographique projeter la conique S suivant un cercle, en rejetant à l' infini soit AB, soit BC, soit CA. Si chaque fois on détermine la position du point i sur chaque circonférence comme on l'a indiqué plus haut, puis qu'on fasse la transformation inverse, on obtiendra nécessairement le même point I de la conique S, au moyen des constructions transformées. On est ainsi conduit au résultat suivant :

On donne trois points A, B, C *sur une conique* S, *et un point* D *sur une droite* L. *Soient* α, β, γ *les points d'intersection des droites* BC, CA *et* AB *avec la droite* L, *et* A′, B′, C′ *les seconds points de rencontre des droites* AD, BD, CD *avec la conique* S. *Les trois droites* αA′, βB′, γC′ *se coupent en un même point* I *situé sur la conique* S.

Nous voyons ainsi qu' à un système de quatre points définis, comme nous l'avons fait, par rapport à une conique S, et à une droite L, correspond un point ·I parfaitement determiné de la conique. Si donc inversement au lieu de se donner la droite L, on se donne le point I, la droite L sera parfaitement déterminée, et passera par les quatre points α, β, γ et D.

On pent donc énoncer le théorème suivant :

Théorème : *Par un point* D *du plan d'une conique, on mène trois sécantes* AA′, BB′, CC′, *et l'on prend un point* I *sur la courbe. Si l'on prolonge chacun des côtés du triangle* ABC *jusqu' à son intersection avec la droite qui joint le point* I *à la deuxième extrémité de la sécante aboutissant au sommet opposé, on obtient trois points, situés sur une ligne droite qui passe par le point* D.

Ce théorème conduit à de nombreuses conséquences. Tout d'abord il donne une construction par points d'une conique définie par cinq points, cette construction déterminant chaque fois deux nouveaux points de la courbe.

Considérons par exemple la conique définie par les cinq points I, A, B, A′, B′ ; joignons AA′ et BB′ qui se coupent au point D, puis menons par le point I une droite quelconque IC, elle rencontre la courbe en un second point inconnu C. Soit γ le point d'intersection de IC avec A′B′ : la droite Dγ est la droite L. Si donc nous prolongeons IA et IB jusqu'aux points α et β où ces droites rencontrent L, il suffira de joindre αB′ et βA′ pour obtenir à leur intersection le point C′ ; le point C sera par suite sur la droite C′D.

On a ainsi autant de couples de points que l'on veut.

Il est facile d'avoir la tangente à la conique en l'un des cinq points donnés. Pour cela, il suffit d'imaginer que deux points de la courbe sont confondus en ce point, que nous désignerons pour cette raison par (A′, B′). Associons lui l'un des autres points donnés, où nous supposerons aussi deux points confondus, soit (A, B). Les droites (A, B) (A′, B′) et CC′ se coupent en D ; I (A, B) et C′ (A′, B′) se coupent au point (α, β). La droite L est déterminée par

les point D et (α, β). Si l'on joint le point γ, où IC rencontre cette droite, au point (A′, B′) on obtient la tangente à la conique en ce point.

Avant de signaler d'autres applications, remarquons que le théorème de Pascal sur l'hexagone inscrit à une conique est une conséquence directe du théorème précédent. Considérons en effet l'hexagone inscrit dont les sommets consécutifs sont A′, C′, C, I, B, B′. Les côtés opposés se coupent aux points D, β et γ situés sur la droite L.

Il est assez naturel de chercher si, inversement, le théorème de Pascal pourrait conduire au théorème que nous venons d'établir.

Les travaux de Plücker, Cayley, Kirkman et autres géomètres contemporains ont établi un très grand nombre de résultats relatifs aux hexagones obtenus en joignant de toutes les manières possibles six points d'une conique, et en construisant les droites de Pascal correspondantes. La plupart sont enoncés dans une note qui termine l'ouvrage de Salmon sur les sections coniques. Mais toutes ces recherches ont pour point de départ la figure formée par *six points déterminés de la courbe* et six seulement. Nous n'y voyons pas de conséquences, au moins immédiates, relatives à ce septième point de la conique, dont l'introduction constitue en quelque sorte le caractère du théorème précédent.

Toutefois, il est facile d'obtenir ce théoréme par l'application répétée du théorème de Pascal à différents hexagones ayant pour sommets six des septs points considerés. Voici l'une des manières les plus simples d'y arriver : considérons les trois hexagones

(1) IA′ABCC′
(2) IB′BCAA′
(3) IC′CABB′

ayant un sommet commun I, chacun des autres se déduisant de ceux de l'hexagone précédent par une permutation circulaire effectuée sur les lettres. Les trois Pascals correspondants sont, en prenant la notation connue

$$\frac{\left\{\begin{matrix}\mathrm{IA'} & \mathrm{A'A} & \mathrm{AB}\\ \mathrm{BC} & \mathrm{CC'} & \mathrm{IC'}\end{matrix}\right\}}{\alpha \quad \mathrm{D} \quad \gamma} \qquad \frac{\left\{\begin{matrix}\mathrm{IB'} & \mathrm{B'B} & \mathrm{BC}\\ \mathrm{CA} & \mathrm{AA'} & \mathrm{IA'}\end{matrix}\right\}}{\beta \quad \mathrm{D} \quad \alpha} \qquad \frac{\left\{\begin{matrix}\mathrm{IC'} & \mathrm{C'C} & \mathrm{CA}\\ \mathrm{AB} & \mathrm{BB'} & \mathrm{IB'}\end{matrix}\right\}}{\gamma \quad \mathrm{D} \quad \beta}$$

Chacune des trois droites a deux points communs avec chacune des deux autres ; elles coïncident donc—les quatres points α, β, γ, D sont donc en ligne droite.

Reprenons maintenant le raisonnement qui nous a conduit du théorème primitif au théorème de Pascal ; il va nous permettre de donner à celui-ci une extension remarquable.

Nous avons fait abstraction ici de la droite AA′ qui joint le point D au sommet de l'hexagone compris entre B′ et C′. Mais rien ne distingue, au point de vue projectif, ce sommet du sommet I compris entre B et C. Si donc on mène la droite DI, qui rencontre la conique en un second point I′, la droite IA′ et la droite BC iront aussi se couper sur la droite $D\alpha\beta\gamma$.

Remarquons maintenant que le point D peut simplement être défini comme le point de rencontre de deux côtés opposés de l'hexagone inscrit à la conique. On peut donc appliquer sans modification le résultat précédent aux deux autres points β et C, ce qui donnera quatre nouveaux points situés sur la droite L. En résumé nous obtenons neuf points en ligne droite, parmi lesquels se trouvent les trois points du théorème de Pascal, et l'on peut énoncer :

Si l'on inscrit un hexagone à une conique, les côtés opposés se coupent deux à deux en trois points en ligne droite. La droite qui joint le point de concours de deux côtés à l'un des deux sommets non situés sur ces côtés rencontre la courbe en un autre point. Si l'on joint ce point au sommet opposé au premier, la droite qui joint les deux sommets voisins de celui-ci rencontre la précédente en un point de la droite de Pascal. On obtient ainsi trois systèmes de trois points situés sur cette droite.

Il suffit de particulariser les conditions du théorème précédent pour en déduire l'énoncé d'autant de propriétés particulières.

Considérons par exemple une parabole, deux cordes parallèles à la tangente au sommet, AA′ et BB′ ; si l'on mène les droites $B\beta$, $A\alpha$ parallèle à l'axe $S\gamma$, elles rencontrent respectivement les cordes A′S, B′S en des points β et α situés sur la perpendiculaire à l'axe au point d'intersection γ de celui-ci avec la corde B′A′.

Prenons maintenant une hyperbole, et supposons que le point I du théorème soit à l'infini sur la courbe ainsi que le point A, le point D étant lui-même infiniment éloigné dans une direction déterminée ; la droite AI devient une asymptote de la courbe, et AA′ est rejetée à l'infini. On est alors conduit à la propriété suivante : Soient deux sécantes parallèles rencontrant l'hyperbole aux points BB′ et CC′. Les cordes BC et B′C′ rencontrent les asymptotes de la courbes en des points situés deux à deux sur deux droites parallèles aux

sécantes données. Chacune de ces droites passe aussi par un des points d'intersection des parallèles aux asymptotes menées par les extrémités de chacune des cordes CB′ et BC′.

On peut appliquer cette propriété à la construction des hyperboles.

La méthode de transformation des figures par polaires réciproques permet de déduire des théorèmes précédents autant de théorèmes corrélatifs sur les tangentes aux coniques.

Note on the regular solids.

By Professor STEGGALL.

The usual methods of proving the existence of regular polyhedra, as given in Wilson and in Todhunter, appear to most students somewhat difficult. It seemed worth while trying, therefore, whether a simpler or more direct proof could not be obtained. The following note shows how this may be done.

The problem is to distribute p points uniformly on a sphere. Suppose that they are arranged in n-sided polygons, and that each angle is $2\pi/m$, then the number of polygons $= pm/n$, and their total area is $pm(2n\pi/m + 2\pi - n\pi)/n = 4\pi$, the radius of the sphere being unity.

Hence $$\frac{2}{m} + \frac{2}{n} - 1 = \frac{4}{pm}$$

of which the only admissible solutions are $m = n = 3$; $m = 3$, $n = 4$ or 5; $m = 4$ or 5, $n = 3$; giving all the five figures.

A certain cubic connected with the triangle.

By Professor STEGGALL.

In examining some of the lines that occur in connection with the recent geometry of the triangle, the cubic whose equation in trilinear co-ordinates is

$$abc(\beta^2 - \gamma^2) + \beta ca(\gamma^2 - \alpha^2) + \gamma ab(\alpha^2 - \beta^2) = 0$$

incidentally appeared, and it seems worth noting the very large number of special points it passes through. These are the vertices, the mid-points of the sides, the inscribed and escribed centres, the circumcentre, the orthocentre, the centroid and the symmedian points, or fourteen in all.

The tangents to this cubic are also interesting: that at a vertex is a symmedian line; those at the inscribed and escribed centres pass through the centroid; that at the centroid through the symmedian point.

Two other cubics also appeared, determined thus:—Let PD, PE, PF be drawn perpendicular to BC, CA, AB; then if AD, BE, CF meet in Q, we have the following loci of P and Q respectively in trilinear co-ordinates:

$$\begin{aligned} &\alpha(\beta^2 - \gamma^2)(\cos A - \cos B\cos C) \\ &+ \beta(\gamma^2 - \alpha^2)(\cos B - \cos C\cos A) \\ &+ \gamma(\alpha^2 - \beta^2)(\cos C - \cos A\cos B) = 0\,; \qquad (1) \end{aligned}$$

and

$$\begin{aligned} &\alpha\cos A(b^2\beta^2 - c^2\gamma^2) \\ &+ \beta\cos B(c^2\gamma^2 - a^2\alpha^2) \\ &+ \gamma\cos C(a^2\alpha^2 - b^2\beta^2 \qquad = 0. \qquad (2) \end{aligned}$$

These two cubics agree with the first cubic discussed when $a = b = c$, as is otherwise clear: the cubic (1) passes through the vertices, the inscribed and escribed centres, the circumscribing centre, the orthocentre.

Edinburgh Mathematical Society.

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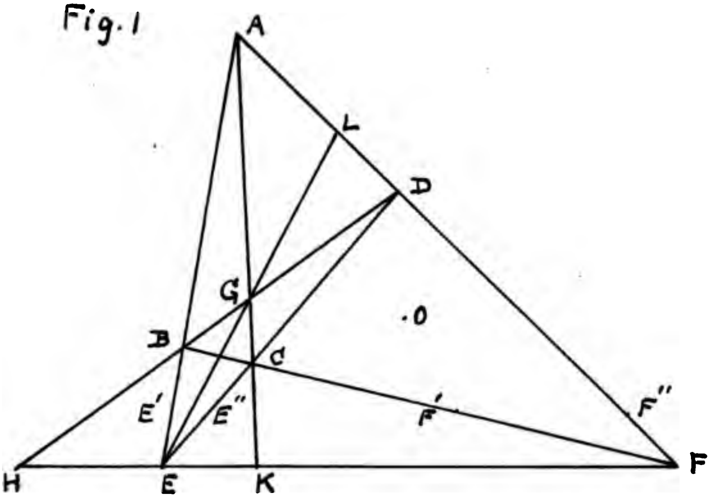
Fig. 1



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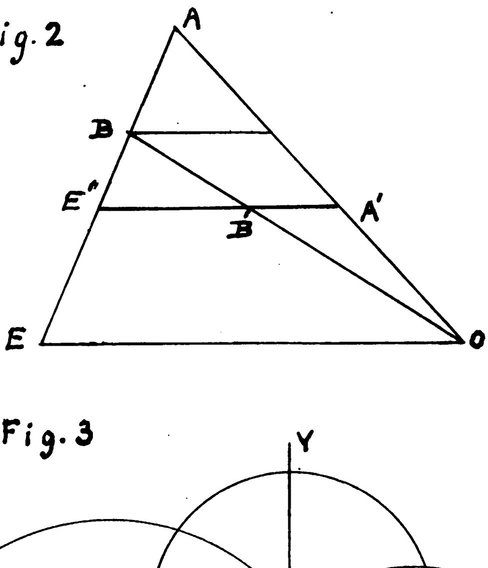

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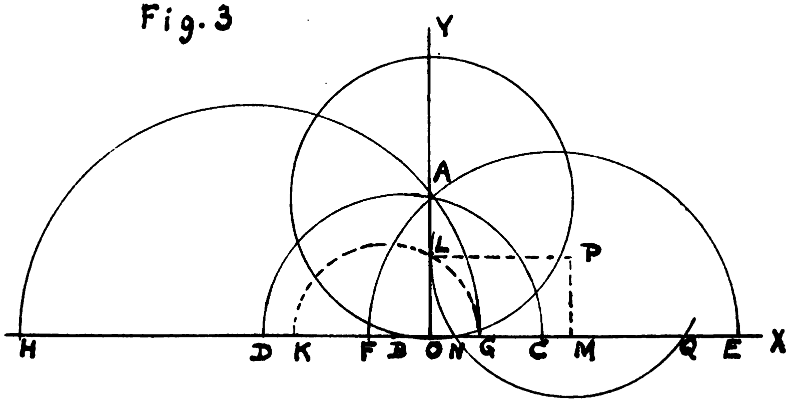

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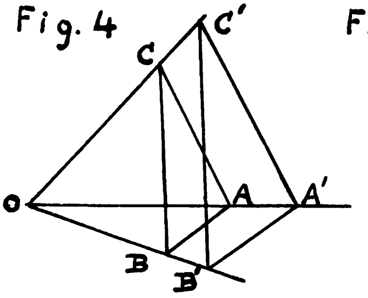

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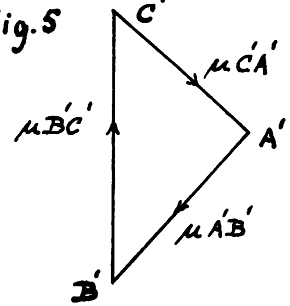

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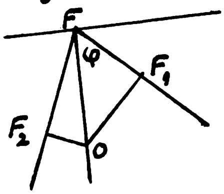

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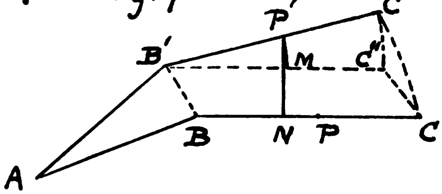

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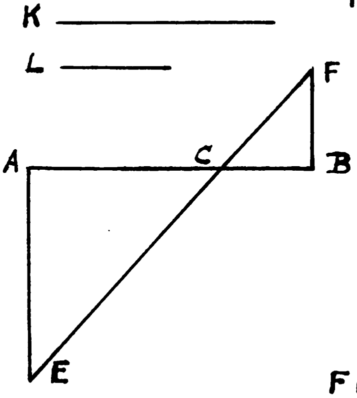

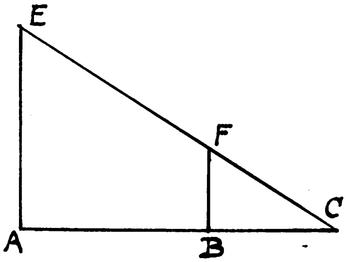

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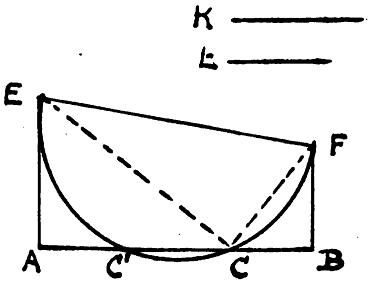

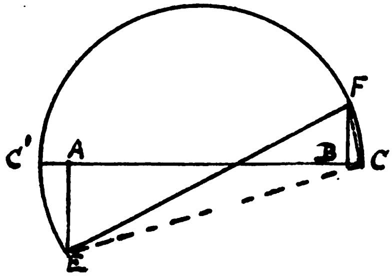

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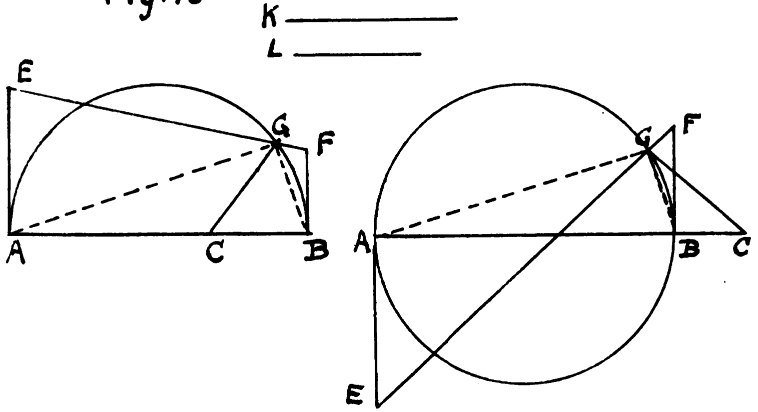

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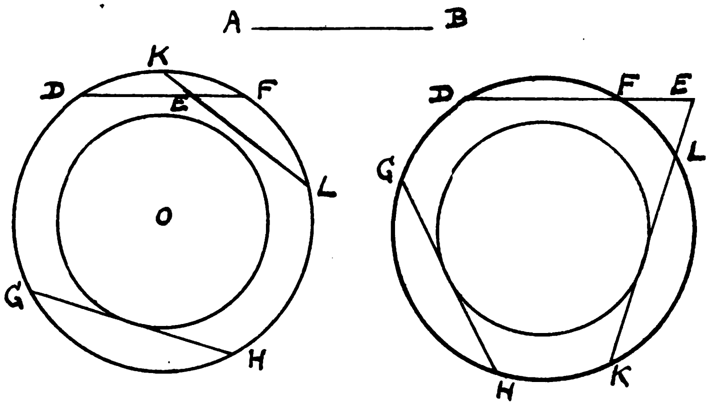

Fig. 12